Chapter 1

Optimization Problems in Graphs and Networks

1.1 Directed Graphs

Definition. *A **directed graph** (**digraph**) is a pair $D = (V, A)$ with a finite set V and a set*

$$A \subseteq V \times V \setminus \{(v, v) : v \in V\}$$

of ordered elements of pairs from V.

- $\mathrm{V}(D) := V$ *: set of **nodes** of D*
- $\mathrm{A}(D) := A$ *: set of **arcs** of D*

For $a = (v, w) \in A$:

- *Alternative notation: $a = vw$*
- *v is the **tail** of a.*
- *w is the **head** of a.*
- *a is an **out-arc** of v.*
- *a is an **in-arc** of w.*
- *w is an **out-neighbor** of v.*
- *v is an **in-neighbor** of w.*

[1]

Definition. *A **directed path** is a digraph P with node set*

$$\mathrm{V}(P) = \{v_0, \ldots, v_k\}$$

($k \geq 0$) with pairwise different $v_0, \ldots, v_k$ and arc set

$$\mathrm{A}(P) = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\};$$

*its **length** is k. P is an s-t-**path**, if $v_0 = s$ and $v_k = t$ hold.*

Definition. *A **directed cycle** is a digraph C with node set*

$$\mathrm{V}(C) = \{v_1, \ldots, v_k\}$$

($k \geq 2$) with pairwise different $v_1, \ldots, v_k$ and arc set

$$\mathrm{A}(C) = \{(v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, v_1)\};$$

*its **length** is k.*

[2]

Definition. *A digraph D' is a* ***subdigraph*** *of a digraph D, if*

$$\mathrm{V}(D') \subseteq \mathrm{V}(D)$$

and

$$\mathrm{A}(D') \subseteq \mathrm{A}(D)$$

hold; if we have

$$\mathrm{V}(D') = \mathrm{V}(D)\,,$$

then we call D' a ***spanning subdigraph*** *of D.*

[3]

Definition. *Let D be a digraph. A* ***path / cycle in*** *D is a subdigraph H of D that is a directed path / cycle.*

Remark.

- *A spanning path / cycle in a digraph D is called a* ***Hamiltonian path / cycle*** *in D.*
- *A subset $F \subseteq \mathrm{A}(D)$ of the arcs of a digraph D is often identified with the subdigraph $(\mathrm{V}(F), F)$ of D, where $\mathrm{V}(F) \subseteq \mathrm{V}(D)$ is the set of all nodes that are the tail or the head of at least one of the arcs in F.*

[4]

Definition. *Let A be a finite set and $B \subseteq A$. For $v \in \mathbb{R}^A$ define*

$$v(B) := \sum_{b \in B} v_b\,.$$

[5]

Definition. *The* ***(directed) shortest path problem*** *is to find for a given digraph D, a cost vector $c \in \mathbb{R}^{\mathrm{A}(D)}$ and $s, t \in \mathrm{V}(D)$ an s-t-path $P \subseteq \mathrm{A}(D)$ with minimal costs $c(P)$.*

[6]

Definition. *The* ***(asymmetric) traveling salesman problem*** *is to find for a given digraph D and a cost vector $c \in \mathbb{R}^{\mathrm{A}(D)}$ a Hamiltonian cycle $K \subseteq \mathrm{A}(D)$ with minimal costs $c(K)$.*

[7]

Definition. *The* ***(asymmetric) Hamiltonian shortest path problem*** *is to find for a given digraph D, a cost vector $c \in \mathbb{R}^{\mathrm{A}(D)}$, and $s, t \in \mathrm{V}(D)$ a Hamiltonian s-t-path $H \subseteq \mathrm{A}(D)$ with minimal costs $c(H)$.*

[8]