1.2 Undirected Graphs

Definition. *An undirected **graph** is a pair $G = (V, E)$ with a finite set V and a set $E \subseteq \binom{V}{2}$ of two-element subsets of V.*

- $\mathrm{V}(G) := V$ *: set of **nodes** of* G
- $\mathrm{E}(G) := E$ *: set of **edges** of* G

For $e = \{v, w\} \in E$:

- *Alternative notation: $e = vw (= wv)$*
- *v and w are **end nodes** of e.*
- *v and w are **incident** to e.*
- *v and w are **neighbors** of each other.*
- *v and w are **adjacent**.*

[14]

Definition. *For a graph G and a node $v \in \mathrm{V}(G)$ we call*

$$\delta(v) := \delta_G(v) := \delta_G(\{v\})$$

*the **star** of v and $\deg(v) := \deg_G(v) := |\delta_G(v)|$ the **degree** of v.*

[15]

Remark. *In every graph the number of nodes with odd degrees is even.*

[16]

Definition. *A **path** is a graph P with node set*

$$\mathrm{V}(P) = \{v_0, \ldots, v_k\}$$

($k \geq 0$) with pairwise different $v_0, \ldots, v_k$ and edge set

$$\mathrm{E}(P) = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\};$$

*its **length** is k. P is an s-t-**path**, if $v_0 = s$ and $v_k = t$ (or $v_0 = t$ and $v_k = s$) hold.*

Definition. *A **cycle** is a graph C with node set*

$$\mathrm{V}(C) = \{v_1, \ldots, v_k\}$$

($k \geq 2$) with pairwise different $v_1, \ldots, v_k$ and edge set

$$\mathrm{E}(C) = \{\{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}\};$$

*its **length** is k.*

[17]

Definition. *A graph G' is a* ***subgraph*** *of a graph G, if*

$$V(G') \subseteq V(G)$$

and

$$E(G') \subseteq E(G)$$

hold; if we have

$$V(G') = V(G)\,,$$

then we call G' a ***spanning subgraph*** *of G. If*

$$E(G') = E(G) \cap \binom{U}{2}$$

holds for some subset $U \subseteq V(G)$ of nodes then G' is the ***subgraph of*** G ***induced by*** *U.*

[18]

Definition. *Let G be a graph. A* ***path / cycle in*** G *is a subgraph H of G that is a path / cycle.*

Remark.

- *A spanning path / cycle in a graph G is called a* ***Hamiltonian path / cycle*** *in G.*
- *A subset $F \subseteq E(G)$ of the edges of a graph G is often identified with the subgraph $(V(F), F)$ of G, where $V(F) \subseteq V(G)$ is the set of all nodes that are an end node of at least one of the edges in F.*

[19]

Remark. *Similarly to the directed cases one defines the* ***(undirected) shortest path problem****, the* ***(symmetric) traveling salesman problem****, and the* ***(undirected) Hamiltonian shortest path problem****.*

Definition. *A graph G is called **connected**, if for every two nodes $s, t \in \mathrm{V}(G)$ there is an s-t-path in G.*

Definition. *The **components** (also: **connected components**) of a graph are its maximal connected induced subgraphs.*

[20]

Definition. *A graph is called **acyclic** or a **forest**, if it does not contain any cycle (i.e., if none of its subgraphs is a cycle). A connected forest is called a **tree**.*

[21]

Definition. *The **spanning tree problem** is to find for a given graph G and a cost vector $c \in \mathbb{R}^{\mathrm{E}(G)}$ a spanning tree $T \subseteq \mathrm{E}(G)$ with minimal costs $c(T)$.*

[22]

Theorem 1. *The following statements are pairwise equivalent for a graph G:*

1. *G is a tree.*
2. *For any pair $s, t \in \mathrm{V}(G)$ ($s \neq t$) of nodes there is exactly one s-t-path in G.*
3. *G is connected, but for every edge $e \in \mathrm{E}(G)$ of G the subgraph $(\mathrm{V}(G), \mathrm{E}(G) \setminus \{e\})$ is not connected.*
4. *G is acyclic, but for every non-edge $\bar{e} \in \binom{\mathrm{V}(G)}{2} \setminus \mathrm{E}(G)$ of G the graph $(\mathrm{V}(G), \mathrm{E}(G) \cup \{\bar{e}\})$ contains a cycle.*

[23]

Corollary 2.

1. *Every connected graph contains a spanning tree.*
2. *Every tree T (with $|\mathrm{V}(T)| \geq 1|$) has exactly $|\mathrm{V}(T)| - 1$ edges.*
3. *A graph that has exactly one edge less than it has nodes is a tree if and only if it is acyclic or connected.*

[24]

Definition. *Let G be a graph..*

- *A node set* $S \subseteq \mathrm{V}(G)$ *is called a* ***stable set*** *in G, if* $\mathrm{E}_G(S) = \varnothing$ *holds.*
- *A node set* $K \subseteq \mathrm{V}(G)$ *is called a* ***clique*** *in G, if* $\mathrm{E}_G(K) = \binom{K}{2}$ *holds.*

[25]

Definition. *The* ***stable set problem*** *is to find for a given graph G and a weight vector* $w \in \mathbb{R}^{\mathrm{V}(G)}$ *a stable set* $S \subseteq \mathrm{V}(G)$ *with maximal weight* $w(S)$*.*

[26]

Definition. *The* ***clique problem*** *is to find for a given graph G and a weight vector* $w \in \mathbb{R}^{\mathrm{V}(G)}$ *a clique* $K \subseteq \mathrm{V}(G)$ *with maximal weight* $w(K)$*.*

[27]

Definition. *A graph is* ***bipartite****, if there is a partitioning of its nodes into two stable sets.*

[28]

Theorem 3. *A graph is bipartite if and only if it does not contain any cycle of odd length.*

[29]