Quantifying the error of convex order bounds for truncated first moments

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The concepts of convex order and comonotonicity have become quite popular in risk theory, essentially since Kaas et al. (2000) constructed bounds in the convex order sense for a sum S of random variables without imposing any dependence structure upon it. Those bounds are especially helpful, if the distribution of S cannot be calculated explicitly or is too cumbersome to work with. This will be the case for sums of lognormally distributed random variables, which frequently appear in the context of insurance and finance.

In this article we quantify the maximal error in terms of truncated first moments, when S is approximated by a lower or an upper convex order bound to it. We make use of geometrical arguments; from the unknown distribution of S only its variance is involved in the computation of the error bounds. The results are illustrated by pricing an Asian option. It is shown, that under certain circumstances our error bounds outperform other known error bounds, e.g. the bound proposed by Nielsen and Sandmann (2003).

Keywords: convex order, truncated first moments, stop-loss-premiums, sums of random variables, Asian options

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1 Introduction

In various disciplines sums of lognormally distributed random variables play a crucial role. This is especially so in insurance and stochastic finance, mostly due to the common modelling of stock price processes as geometric Brownian motions.

For example the final payoff S_B of an investment in a portfolio ("basket") consisting of more than one stock is modelled as sum of dependent lognormally distributed random variables, whereas the price of the corresponding so called Basket (Call) option is (up to a constant term) nothing else than a certain truncated first moment $E((S_B - t)^+)$ of S_B. Similarly, the final payoff S_A of a repeated investment (made at consecutive time points) in a stock is a sum of dependent lognormally distributed random variables, whereas $E((S_A - t)^+)$ is (up to a constant term) the price of the corresponding so called Asian (Call) option.[1]

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[1] In both examples the expectation has to be taken with respect to the so called equivalent martingale measure, what is not of importance here, however.

Unfortunately, for the distribution of a sum of lognormally distributed random variables closed-form formulae are not available. Since the simulation of the distribution is often too time-consuming, one has to find another way to approximate the sum distribution. But even in the case of independent summands the application of the central limit theorem is inappropriate due to the typically small number of terms involved in the sum.

To overcome these difficulties, a way out might be to approximate a sum S of lognormally ditributed random variables by a single lognormally distributed random variable L having the same first two moments as S. Indeed, in many practically relevant situations the cumulative distribution function (c.d.f.) of such an L seems to be quite close to the simulated c.d.f. of S. Only recently Dufresne (2004) gave a theoretical justification to do so. He showed that a sum of lognormal random variables will (suitably normalized) converge in distribution to a certain lognormal distribution, if (roughly spoken) the variances of its summands tend uniformly to 0, whereas their correlations stay constant. So one can hope that the approximation is accurate, if the variances of the summands are small. However, an approximation error cannot be quantified, since Dufresne (2004) does not provide convergence rates or bounds for the distributional distance in any sense.

Especially in insurance and finance the concept of convex ordering of random variables seems to be very fruitful. Two random variables with finite first moments are said to be convexly ordered , denoted by $X \leq_{cx} Y$, if

$$E(X-d)^+ \leq E(Y-d)^+ \quad \text{for all } d \in \mathbb{R} \quad \text{and} \quad E(X) = E(Y) \ .$$

X has thus uniformly smaller upper tails than Y. So, if X and Y represent gains (i.e. incoming payments) this means that a risk averse decision maker would prefer the payment Y to the payment X. On the other hand the same decision maker would prefer the payment X to the payment Y, when X and Y represent losses (i.e. payments, one has to settle). Here Y might be seen as "riskier" than X. However, $X \leq_{cx} Y$ always implies $\mathrm{Var}(X) \leq \mathrm{Var}(Y)$.

The relevance of this concept can be easily observed. Consider for example an insurance company, which replaces in its premium calculations the risk X by the risk $Y \geq_{cx} X$, and thereby just follows its usual prudent policy. From this "risk" perspective it seems to be natural to look for lower and/or upper bounds for a sum S of lognormally distributed random variables in the convex order sense.

A general approach to construct convex order bounds for sums of random variables can be found in the review article of Dhaene et al. (2002b): Let $X_1, \ldots, X_n$ be random variables with corresponding c.d.f.'s $F_{X_1}, \ldots, F_{X_n}$ and Z a further random variable. Let $F_{X_i}^{-1}$ denote the usual inverse of F_{X_i} (defined as $F_{X_i}^{-1}(u) := \inf\{x \in \mathbb{R} : F_{X_i}(x) \geq u\}$) and define $F_{X_i|Z}^{-1}(U) := g(Z, U)$, where $g(z, u) := F_{X_i|Z=z}^{-1}(u)$ and $F_{X_i|Z=z}$ denotes the c.d.f. of the conditional distribution of X_i given $Z = z$. Define

$$S^l := \sum_{i=1}^{n} E(X_i|Z) \ , \quad S^u := \sum_{i=1}^{n} F_{X_i|Z}^{-1}(U), \quad S^c := \sum_{i=1}^{n} F_{X_i}^{-1}(U) \ , \tag{1}$$

where U is uniformly distributed on the interval $(0, 1)$ and independent of Z. Then we have

$$S^l \ \leq_{cx} \ \sum_{i=1}^{n} X_i \ \leq_{cx} \ S^u \ \leq_{cx} \ S^c \tag{2}$$

for any random variable Z, cf. Dhaene et al. (2002b) and the references cited therein.

Note, that the bounds S^l and S^u depend on Z; in their representations the conditional distributions of the X_i's given Z are involved. Therefore one should of course in practice choose Z such that one can handle these distributions. If the X_i's are lognormally distributed, this is for instance achieved by choosing Z as linear combination of the logarithmized X_i's. Then the conditional distribution of X_i given Z will be lognormal; all convex order bounds in (1) turn out to be sums of lognormally distributed random variables again.

We now want to summarize some properties of the *TFM-function*[2] $\xi(t) = E[(X-t)^+]$ of a random variable X, which will be used repeatedly in the sequel. Let F_X denote the c.d.f. of X. Then:

(T1) $\xi(t) \geq \max\{0, E(X) - t\} = (E(X) - t)^+$,

(T2) $\lim_{t\to\infty} \xi(t) = 0$, $\lim_{t\to-\infty} \xi(t) - (E(X) - t) = 0$,

(T3) $\xi(t) = E(X) - t \iff F_X(t-) = 0$,

(T4) $\xi(t) = 0 \iff F_X(t) = 1$,

(T5) ξ is convex on $\mathbb{R}$ with left and right derivatives $\xi'_-(t) = F_X(t-) - 1$ and $\xi'_+(t) = F_X(t) - 1$, respectively.

Let us now suppose, we have some lower convex order bound $\underline{S}$ and/or some upper convex order bound $\overline{S}$ for a random variable S (not necessarily a sum of random variables). By definition of the convex order we then have

$$E[(\underline{S} - t)^+] \leq E[(S - t)^+] \leq E[(\overline{S} - t)^+] \quad \text{for all } t \in \mathbb{R} .$$

Taking into account the easy-to-proof properties of TFM-functions from above, we can exemplarily draw the TFM-functions of $\underline{S}, S$ and $\overline{S}$, respectively, see Figure 1. They have the same asymptotes, since the three random variables have the same expectation by the definition of the convex order. Furthermore, they do not cross and decrease with a slope ranging from -1 to 0.

The question arises, how far away the (unknown) curve corresponding to S is from the other two curves. The aim of this paper is to find real functions Δ_l and Δ_u, such that for any $t \in \mathbb{R}$ holds

$$E[(S - t)^+] - E[(\underline{S} - t)^+] \leq \Delta_l(t) \quad \text{and} \quad E[(\overline{S} - t)^+] - E[(S - t)^+] \leq \Delta_u(t) \ . \tag{3}$$

Δ_l resp. Δ_u is thus an upper bound for the error in terms of truncated first moments, when S is approximated by $\underline{S}$ resp. $\overline{S}$. We remark, that we could easily set $\Delta_l(t) = \Delta_u(t) = \Delta^*(t)$, with

$$\Delta^*(t) := E[(\overline{S} - t)^+] - E[(\underline{S} - t)^+] ,$$

but this *trivial bound* might be too large or even not available in practice.

[2] *TFM* stands for *Truncated First Moment.*

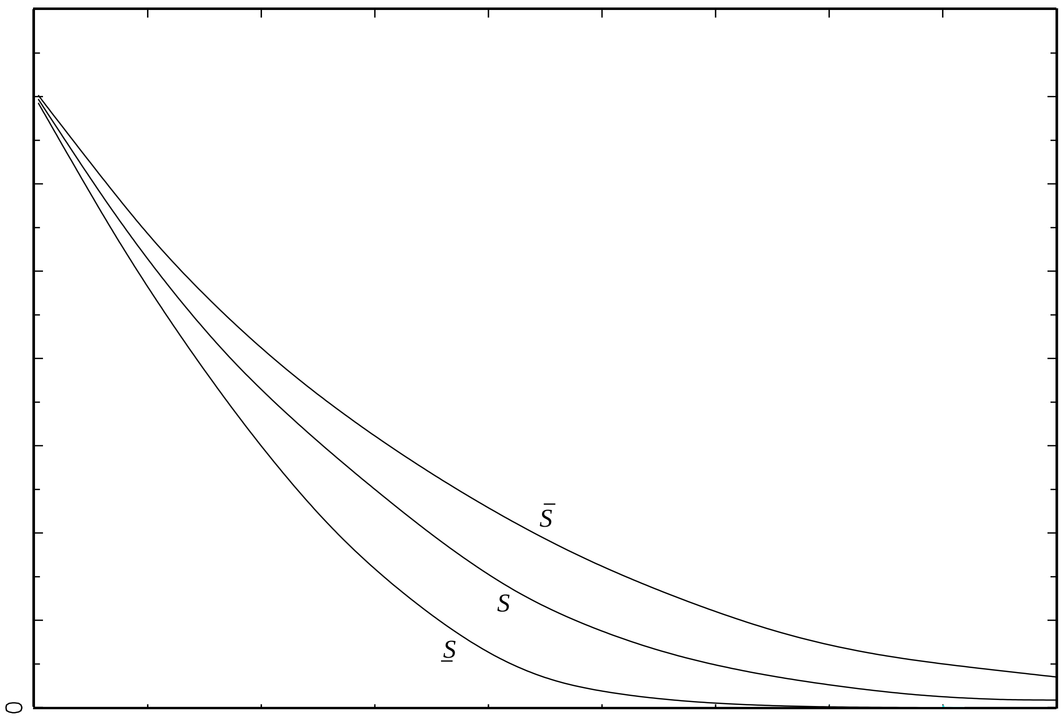

Figure 1: Truncated first moments as functions of t

In Section 2 we will construct the bounds $\Delta_l(t)$ and $\Delta_u(t)$ using geometrical arguments. The construction is based on the following fact, whose proof can be found in Kaas et al. (2001): If X, Y are square-integrable with $E(X) = E(Y)$, then:

$$\text{(T6)} \qquad \int_{-\infty}^{\infty} E[(Y-t)^+] - E[(X-t)^+] \; dt = \frac{1}{2}(\mathrm{Var}(Y) - \mathrm{Var}(X)) \; .$$

To put it into words: the mean difference between the truncated first moments of two random variables having the same expectation equals half the difference of their variances. Note, that this mean difference is nothing else than the area between the corresponding TFM-functions, if $X \leq_{cx} Y$.

In Section 3 the results will be illustrated numerically and graphically. We give upper and lower bounds for the price of an Asian option and compare the results with those of Nielsen and Sandmann (2003).

2 The main result

In this section we consider two convexly ordered random variables X and Y, $X \leq_{cx} Y$, where the distribution of either X or Y is unknown, and construct upper bounds for $\Delta(t) := E[(Y-t)^+] - E[(X-t)^+], \; t \in \mathbb{R}$. We get the setting from Section 1, if we choose $X = S$ and $Y = \overline{S}$, or $X = \underline{S}$ and $Y = S$.

We start with an auxiliary result about the TFM-function of a random variable. At first view it might look somewhat technical, but later on the occuring functions turn out to have a geometrical interpretation.

Lemma 1
Let X be an integrable real random variable with c.d.f. F_X and denote $g : s \mapsto E[(X-s)^+],\ s \in \mathbb{R}$. Then :

(i) For any fixed $t \in \mathbb{R}$ the function

$$q_t : s \mapsto g(s) - g(t) + (s-t), \quad s \in \{F_X > 0\} ,$$

is one-to-one. We have $q_t(\{F_X > 0\}) \supseteq (0, \infty)$.

(ii) The function $g|_{\{F_X<1\}}$ is one-to-one (denote its inverse by g^{-1}) and we have $g(\{F_X < 1\}) = (0, \infty)$.

(iii) For any fixed $t \in \mathbb{R}$ the function

$$h_t : z \mapsto z \cdot [q_t^{-1}(z) - g^{-1}(g(t) + z)], \quad z \in (0, \infty) ,$$

is one-to-one. We have $h_t((0, \infty)) \supseteq (0, \infty)$.

The proof is given in the appendix. Now we are able to state the main result.

Theorem 1
Let X and Y be square-integrable real random variables with $X \leq_{cx} Y$. Let F_Y denote the c.d.f. of Y, M the interior of the interval $\{0 < F_Y < 1\}$, $\varepsilon := Var(Y) - Var(X)$ and

$$\Delta(t) := E[(Y-t)^+] - E[(X-t)^+] \quad \textit{for } t \in \mathbb{R}.$$

Define h_t as above in Lemma 1 and further for all $t \in M$:

$$\Delta_u(t) := \left(\frac{1}{F_Y(t-)} - \frac{1}{F_Y(t) - 1} \right)^{-\frac{1}{2}} \cdot \sqrt{\varepsilon} \quad , \qquad \Delta_l(t) := h_t^{-1}(\varepsilon) \ .$$

Then we have for all $t \in M$:

$$\Delta(t) \ \leq \ \min\{ \ \Delta_u(t) \ , \ \Delta_l(t) \ \} \leq \ \tfrac{1}{2}\sqrt{\varepsilon} \ . \tag{4}$$

If $t \in \mathbb{R} \setminus M$, then $\Delta(t) = 0$.

The preceding theorem provides two bounds, Δ_u and Δ_l, each of them being useful in different practical situations. Consider the situation where one cannot compute $E[(X-t)^+]$ (e.g. if the distribution of X cannot be determined) and therefore need to approximate it by $E[(Y-t)^+]$, where Y is a random variable which dominates X in the convex order sense and whose distribution is known. In doing so, Δ_u provides an upper bound for the error, in whose computation the distribution of X is not involved.

In contrast, Δ_l gives a bound for the complementary situation where Y is replaced by a random variable X, which is dominated by Y in the convex order sense. Consequently, the distribution of Y is not involved in the computation of Δ_l at all.

Proof of Theorem 1.
Throughout the proof we set $g(s) = E[(X - s)^+]$ (as in Lemma 1) and $f(s) = E[(Y - s)^+]$.

The last statement of the theorem can be easily verified. From $X \leq_{cx} Y$ we get $E(X) = E(Y)$ and $g(t) \leq f(t)$ for all $t \in \mathbb{R}$. Obviously, $t \in \mathbb{R} \setminus M$ means that we have either $F_Y(t-) = 0$ or $F_Y(t) = 1$. If $F_Y(t-) = 0$ we get from (T3):

$$f(t) = E(Y) - t = E(X) - t \leq E[(X - t)^+] = g(t) \leq f(t),$$

and thus $\Delta(t) = 0$. If $F_Y(t) = 1$, we get from (T4) that $0 = f(t) \geq g(t) \geq 0$, and thus again $\Delta(t) = 0$.

The idea for the rest of the proof is surprisingly simple. We will inscribe (as large as possible) simple geometric figures between f and g, whose area is known to be bounded by $\varepsilon/2$, cf. (T6). Using triangles having one side length $\Delta(t)$ and utilizing some knowledge about the run of the curves f and g (cf. (T1)-(T5)) then leads to the asserted bounds.

Let from now on $t = t_0 \in M$ be fixed and denote $\delta = \Delta(t_0)$.

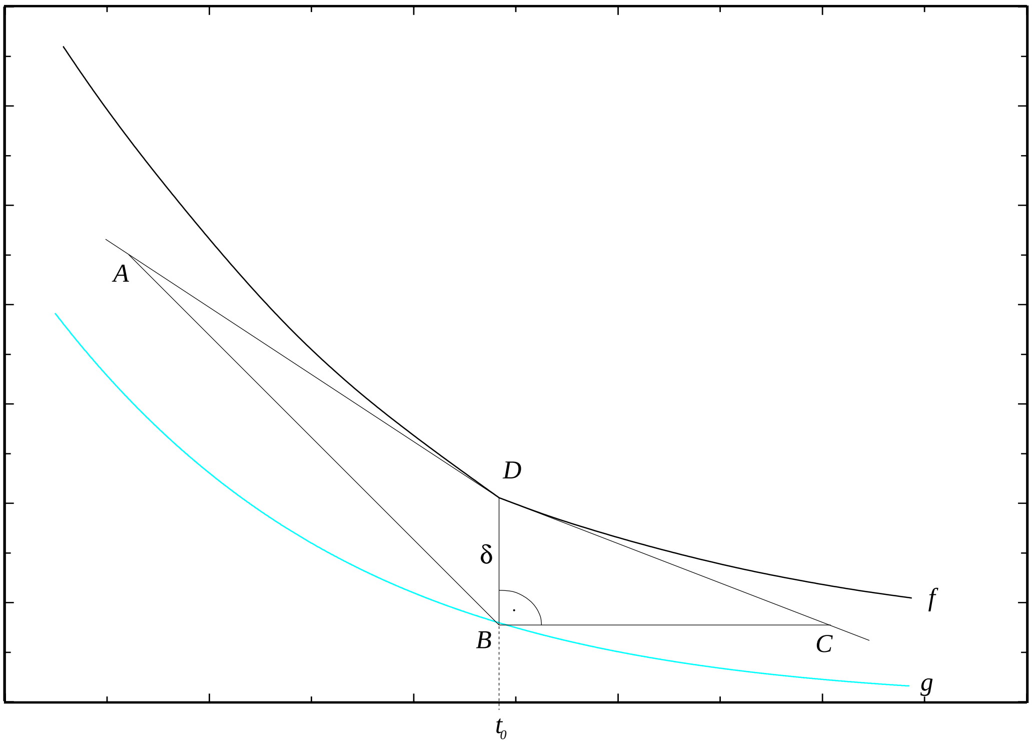

Figure 2: The area between the two curves is an upper bound for the area of the quadrangle $ABCD$

(i): Figure 2 shows how a quadrangle $ABCD$ can be inscribed between the two curves f and g. The area of this quadrangle is a lower bound for $\frac{\varepsilon}{2}$, which is, according to (T6), the area between f and g.

The vertices of the quadrangle are chosen as follows:

$$B = (t_0, g(t_0)), \quad D = (t_0, f(t_0)),$$

and A as the point of intersection of the left tangent on f in t_0 with the straight line $w_1(s) = -s + g(t_0) + t_0$ (which passes through B with slope -1). Here the left (resp. right) tangent on f in t_0 is defined as the straight line with slope $f'_-(t_0)$ (resp. $f'_+(t_0)$) through the point $(t_0, f(t_0))$. Finally, C is the point of intersection of the right tangent on f in t_0 with the constant function $w_2(s) = g(t_0)$.

Using trigonometrical tools we are able to compute the area of the two triangles:

$$\text{area}(\triangle BCD) = -\frac{\delta^2}{2}\frac{1}{f'_+(t_0)} \quad , \quad \text{area}(\triangle ABD) = \frac{\delta^2}{2}\frac{1}{1+f'_-(t_0)} \; . \tag{5}$$

Summing up those two we obtain the desired area of the quadrangle $ABCD$,

$$\frac{\delta^2}{2}\left(\frac{1}{f'_-(t_0)+1} - \frac{1}{f'_+(t_0)}\right),$$

which must be bounded by $\frac{\varepsilon}{2}$ because of (T6). Is thus ε given, we get the following bound for δ,

$$\delta \leq \frac{1}{\sqrt{\frac{1}{F_Y(t_0-)} - \frac{1}{F_Y(t_0)-1}}} \cdot \sqrt{\varepsilon} = \Delta_u(t_0) \; , \tag{6}$$

where we used $f'_-(t_0) = F_Y(t_0-) - 1$ and $f'_+(t_0) = F_Y(t_0) - 1$, cf. (T5).

Note, that the quadrangle $ABCD$ is actually a triangle ABC, if f is differentiable in t_0 or, equivalently, if F_Y is continuous at t_0. The area of the triangle $\triangle ABC$ is in general smaller than that of the quadrangle $ABCD$. This leads in the continuous case to a $\Delta_u(t_0)$, which is – somewhat surprisingly – greater than in the discontinuous case.

Anyway, we can estimate $F_Y(t_0-) \leq F_Y(t_0)$ and $F_Y(t_0) - F_Y^2(t_0) \leq \frac{1}{4}$, which leads us to

$$\Delta_u(t_0) \leq \sqrt{F_Y(t_0) - F_Y^2(t_0)} \cdot \sqrt{\varepsilon} \leq \frac{1}{2}\sqrt{\varepsilon} \; .$$

(ii): Again we inscribe a quadrangle between f and g (see Fig. 3), but this time we will make use of the precise run of the curve g rather than that of f.

To begin with, consider the right triangle $\triangle ABD$ whose vertex points are chosen as follows:

$$A = (g^{-1}(f(t_0)), f(t_0)), \quad B = (t_0, g(t_0)), \quad D = (t_0, f(t_0)) \; .$$

The point A is well defined as a consequence of Lemma 1 *(ii)*.

This triangle lies between f and g, because f is decreasing and g is convex; its area is found to be

$$\text{area}(\triangle ABD) = \frac{\delta}{2}(t_0 - g^{-1}(g(t_0) + \delta)) \; .$$

The point C is found as point of intersection of g with the linear function $m(s) := -s + g(t_0) + \delta + t_0$ (straight line through D with slope -1). This choice of C ensures that the triangle $\triangle BCD$ is located between the two curves f and g, which is due to the convexity of g and the fact that we have $m(s) \leq f(s)$ for $s \geq t_0$. The latter is easily verified, because we have $m(t_0) = f(t_0)$, $m'(s) = -1$ and $f'_+(s) \geq -1$ for all s.

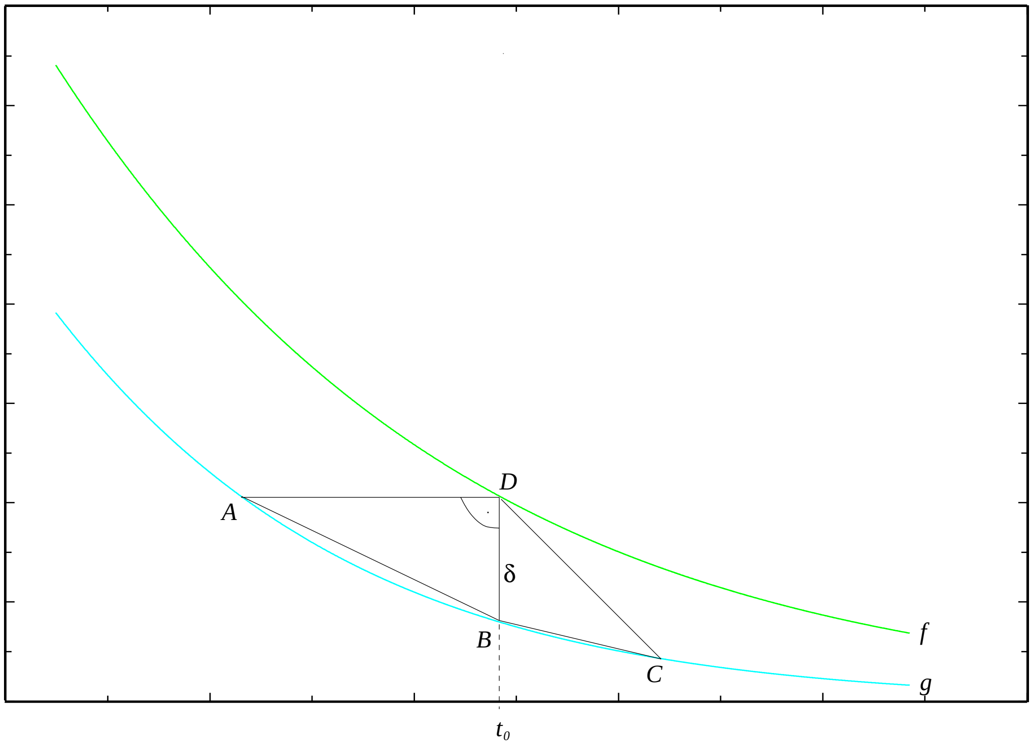

Figure 3: The area between the two curves is an upper bound for the area of the quadrangle $ABCD$

Thus, C is obtained by the solution of the equation $f(s) = m(s)$, i.e. the solution (for s) of

$$g(s) - g(t_0) + (s - t_0) = \delta \ .$$

But this is just $q_{t_0}^{-1}(\delta)$, which in Lemma 1 (i) has been shown to be well defined, and we get

$$C = (\ q_{t_0}^{-1}(\delta), m(q_{t_0}^{-1}(\delta))\ )\ .$$

This enables us to compute the area of the triangle $\triangle BCD$,

$$\text{area}(\triangle BCD) = \frac{\delta}{2}(q_{t_0}^{-1}(\delta) - t_0)\ ,$$

and finally the area of the quadrangle $ABCD$,

$$\frac{1}{2}\,\delta \cdot [\,q_{t_0}^{-1}(\delta) - g^{-1}(\,g(t_0) + \delta\,)\,]\ . \tag{7}$$

With the notations of Lemma 1 we can rewrite (7) as $\frac{1}{2}h_{t_0}(\delta)$, which must be bounded by $\frac{\varepsilon}{2}$ again because of (T6). Note, that $h_{t_0}(\delta)$ can be interpreted geometrically: it is the area of a rectangle, whose one side length is δ and whose other side length is the difference between the abcissas of A and C. The function h_{t_0} is invertible by Lemma 1 (iii), we get thus $\delta \leq h_{t_0}^{-1}(\varepsilon) = \Delta_l(t_0)$.

Combining the results from parts (i) and (ii) of the proof we arrive at

$$\Delta(t_0) = \min\{\Delta_u(t_0), \Delta_l(t_0)\} \leq \frac{1}{2}\sqrt{\varepsilon}\ .$$

Since we have chosen $t_0 \in M$ arbitrarily at the beginning and set $\delta = \Delta(t_0)$, this proves the theorem. □

Consider a sum S of arbitrary random variables once more and suppose that some upper convex order bound $\overline{S}$ resp. lower convex order bound $\underline{S}$ is available. Note, that the computation of the bound $\Delta_u(t)$ for $E[(\overline{S}-t)^+] - E[(S-t)^+]$ involves the distribution of $\overline{S}$. Similarly, the computation of the bound $\Delta_l(t)$ for $E[(S-t)^+] - E[(\underline{S}-t)^+]$ involves the distribution of $\underline{S}$. It arises the question, whether the distribution of the convex order bounds in (1) can be (easily) determined. The answer is *Yes* at least for the upper convex bounds S^c and S^u (although the computation of the latter involves a (numerical) integration), and that relies on the comonotonicity property[3] of the summands of these convex order bounds. However, the answer might be *No* for the lower convex order bound S^l.

3 Application: Numerical and graphical illustration

In this section we illustrate by pricing Asian options, how the *geometrical* bounds Δ_u, Δ_l from Theorem 1 perform in comparison with other bounds based on convex ordering.

The section is organized as follows: in Subsection 3.1 we state the underlying model for the whole section and specify, which convex order bounds for the unknown distribution are considered. In Subsection 3.2 we shortly explain, how to compute another upper bound for $E[(S-t)^+] - E[(S^l-t)^+]$ following Nielsen and Sandmann (2003). This is done, because their bound has been found to perform quite well and should be compared with our corresponding bound Δ_l. Finally, Subsection 3.3 contains the numerical and graphical results.

3.1 The setting

We consider a sum S of lognormally distributed random variables, i.e.

$$S = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} e^{Y_i}\ , \tag{8}$$

where $Y = (Y_1, \ldots, Y_n)$ is multivariate $N(\mu, \Sigma)$-distributed, with $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ and $\Sigma = ((\sigma_{ij}))_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}$ non-negative definite.

Throughout this section the geometrical bounds Δ_u and Δ_l will be computed on the basis of the convex order bounds S^c and S^l of S, which were stated in (1). Hence the geometrical bound Δ_u will be computed for $E[(S^c-t)^+]-E[(S-t)^+]$, choosing $X = S$

[3]For definition of comonotonicity and related results the reader is referred to Kaas et al. (2000). Of interest is here is the following nice property: If $X_1, \ldots, X_n$ are comonotonic, then any quantile of $\sum_{i=1}^n X_i$ is the sum of the corresponding quantiles of the X_i's and any truncated first moment of $\sum_{i=1}^n X_i$ is (basically) the sum of certain truncated first moments of the X_i's.

and $Y = S^c$ in Theorem 1, whereas the geometrical bound Δ_l will be computed for $E[(S-t)^+] - E[(S^l-t)^+]$, choosing $X = S^l$ and $Y = S$ in Theorem 1.

Kaas et al. (2000) gave explicit expressions for the convex order bounds S^c and S^l, which can be easily verified.

For the upper convex order bound holds

$$S^c \stackrel{d}{=} \sum_{i=1}^{n} \exp\left\{\mu_i + \sigma_i \Phi^{-1}(U)\right\} \ ,$$

where we abbreviated $\sigma_i^2 := \sigma_{ii}$; U is uniformly distributed on the interval $(0,1)$. Due to the comonotonic structure of S^c any truncated first moment of it can be easily computed.

If the conditioning random variable Z is choosen as a linear combination of the Y_i's, $Z = \sum_{i=1}^n c_i Y_i$, we get for the lower convex order bound:

$$S^l \stackrel{d}{=} \sum_{i=1}^{n} \exp\left\{\mu_i + r_i \sigma_i \Phi^{-1}(U) + \frac{1}{2}(1-r_i^2)\sigma_i^2\right\} \ , \tag{9}$$

where $r_i := \mathrm{Corr}(Y_i, Z)$. S^l will be a sum of comonotonic random variables, if $r_i \geq 0$ for all i. It should be noted, that in the Asian option setting from Subsection 3.3 the latter will be achieved by *any* such Z with positive coefficients.

Following Vanduffel et al. (2005), we choose the conditioning random variable Z such that a first order approximation of $\mathrm{Var}(S^l)$ is maximized:

$$Z = \sum_{i=1}^{n} e^{\mu_i + \sigma_i^2/2} \ Y_i \ . \tag{10}$$

The resulting S^l will then be close to S in the convex order sense, i.e. in terms of (T6).

In the sequel the bound Δ_u will be compared with the trivial bound $\Delta^*(t) = E[(S^c - t)^+] - E[(S^l - t)^+]$, just like the bound Δ_l, but the latter will additionally be compared with another bound established by Nielsen and Sandmann (2003).

3.2 The Nielsen-Sandmann bound

Based on preliminary work of Rogers and Shi (1995) they constructed an upper bound for

$$E[(S-t)^+] - E[(S^l-t)^+] \ , \tag{11}$$

where S is the payoff of an Asian option. But their approach is actually fairly general: for a sum S of *arbitrary* random variables an upper bound for (11) is given by

$$\Delta_{NS}(t) := \frac{1}{2}\left(E[\mathrm{Var}(S|Z) 1_{\{Z<d(t)\}}]\right)^{\frac{1}{2}} \left(P(Z<d(t))\right)^{\frac{1}{2}} \ , \tag{12}$$

provided that there exists a constant $d(t) \in \mathbb{R}$ such that $\{Z \geq d(t)\} \subset \{S \geq t\}$.

In our lognormal setting we arrive at

$$\Delta_{NS}(t) = \frac{1}{2}\sqrt{\Phi(d^*(t)) \sum_{i=1}^{n}\sum_{j=1}^{n}(e^{\sigma_{ij}} - e^{r_i r_j \sigma_i \sigma_j})e^{\mu_i+\mu_j+\frac{1}{2}(\sigma_i^2+\sigma_j^2)}\Phi(d^*(t) - r_i\sigma_i - r_j\sigma_j)}\ , \tag{13}$$

where we abbreviated $d^*(t) := (d(t) - E(Z))/\sqrt{\mathrm{Var}(Z)}$. In the special setting, where S is the payoff of a Basket option already Deelstra et al. (2004) gave a formula for Δ_{NS}.

We will now construct the values $d(t),\ t \in \mathbb{R}$, for our choice (10) of the conditioning random variable Z. For that purpose we make use of the estimate $e^x \geq 1 + x$, which holds for all $x \in \mathbb{R}$, but is the better the closer x is to zero. This yields

$$S = \sum_{i=1}^{n} e^{Y_i} = \sum_{i=1}^{n} e^{\mu_i+\sigma_i^2/2}e^{Y_i-\mu_i-\sigma_i^2/2} \geq \sum_{i=1}^{n} e^{\mu_i+\sigma_i^2/2}(1 - \mu_i - \tfrac{\sigma_i^2}{2}) + \sum_{i=1}^{n} e^{\mu_i+\sigma_i^2/2}Y_i\ ,$$

and the last summand on the r.h.s. is just Z. Note, that this estimate is the better the smaller the variances σ_i^2 are. Obviously,

$$d(t) = t - \sum_{i=1}^{n} e^{\mu_i+\sigma_i^2/2}(1 - \mu_i - \tfrac{\sigma_i^2}{2}) \tag{14}$$

fulfills $\{Z \geq d(t)\} \subset \{S \geq t\}$ for all $t \in \mathbb{R}$. Note, that $d^*(t) \to \infty$ for $t \to \infty$, which leads to

$$\lim_{t\to\infty} \Delta_{NS}(t) = \frac{1}{2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(e^{\sigma_{ij}} - e^{r_i r_j \sigma_i \sigma_j})e^{\mu_i+\mu_j+\frac{1}{2}(\sigma_i^2+\sigma_j^2)}} = \frac{1}{2}\sqrt{\mathrm{Var}(S) - \mathrm{Var}(S^l)}\ .$$

Since $\Delta_{NS}(t)$ is increasing (recall the definition of Δ_{NS} in (12) and take into account (14)), the above limit must be an upper bound for it and thus for $E[(S - t)^+] - E[(S^l - t)^+]$, but that has been already established in Theorem 1. To make the above limit as small as possible one obviously has to choose Z such that $\mathrm{Var}(S^l)$ gets as large as possible, which legitimizes the choice (10) once more.

We mention that in the literature other proposals for the choice of the conditioning random variable Z have also been made: Kaas et al. (2000) choose it as a linear transformation of a first order approximation to S and Nielsen and Sandmann (2003) (in the context of pricing Asian options) as the standardized logarithm of the geometric average $G = (\prod_{i=1}^{n} X_i)^{1/n}$ of the X_i's. Also for those choices of Z exist $d(t) < \infty$ for all $t \in \mathbb{R}$ with the desired property.

3.3 Pricing Asian options

Within the Black-Scholes setting we consider a forward starting European-style discrete arithmetic Asian call option with fixed strike having the same characteristics as in Nielsen and Sandmann (2003) and Vanmaele et al. (2006): an initial stock price $s_0 = 100$, a (continuously compounded) interest rate of $r_Y = 0.04$ per year (corresponding to $r = 0.04/12$ per month), a yearly volatility of $\sigma_Y = 0.25$ (corresponding to a monthly volatility of $\sigma = 0.25/\sqrt{12}$), a maturity of $T = 36$ months and an averaging period of 36 months, where the averaging is done monthly.

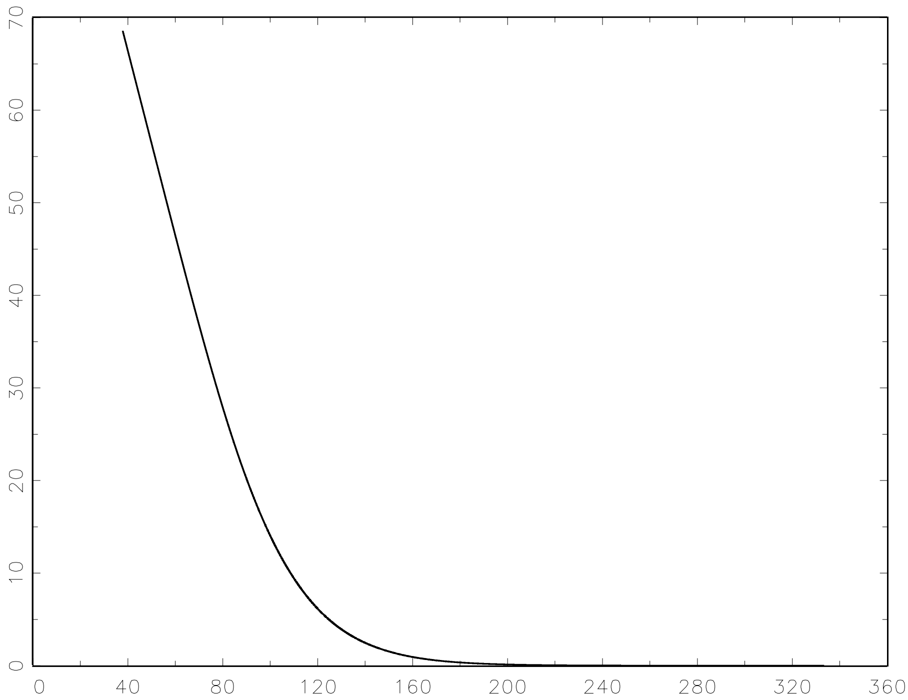

Figure 4: Expected payoff of the option as function of the strike t

If $(A_t)_{t>0}$ denotes a geometric Brownian motion with drift r and volatility σ and

$$S = \frac{1}{n}\sum_{i=1}^{n} A_{T-(i-1)} \ ,$$

then the expected payoff of the option with strike t is $E[(S-t)^+]$ and the corresponding price is obtained through discounting. Note, that S can be written as in (8), with

$$\mu_i = \ln s_0 - \ln n + (r - \tfrac{\sigma^2}{2})(T-i+1) \ , \quad \sigma_{ij} = \sigma^2 \min\{T-i+1, T-j+1\} \ .$$

To spare the reader with large tables we prefer to show some graphics. In figure 4 the function $t \mapsto E[(S-t)^+]$ is plotted as a result of a simulation based on 100000 paths to get an impression of the magnitude of the expected payoff.

In figure 5 we plotted for $E[(S^c-t)^+] - E[(S-t)^+]$ the geometrical bound Δ_u and the trivial bound Δ^*, in addition we indicated by a straight line the uniform bound $\frac{1}{2}\sqrt{\text{Var}(S^c) - \text{Var}(S)}$. It can be seen, that the geometrical bound performs for virtually all t worse than the trivial bound. This is due to a quite large difference (263.09) of the variances. The maximum of Δ_u (verify that this is achieved for any 0.5-quantile of S^c) is attained somewhere around $E(S) = 106.42$, since the distribution of S^c is not very skewed. Although Δ_u performs worse than Δ^*, recall that the latter might not be available in other settings.

In figure 6 we plotted for $E[(S-t)^+] - E[(S^l-t)^+]$ the geometrical bound Δ_l, the trivial bound Δ^*, the Nielsen-Sandmann bound Δ_{NS}; again we indicated the uniform bound $\frac{1}{2}\sqrt{\text{Var}(S) - \text{Var}(S^l)}$ by a straight line. This time the difference of

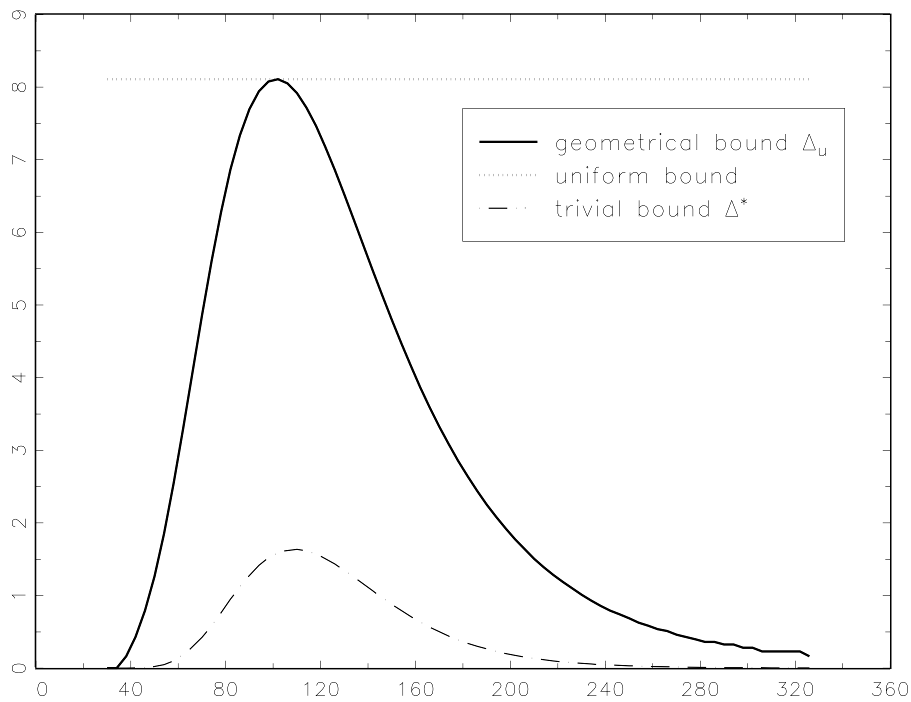

Figure 5: Upper bounds for the deviation of the expected payoff from its upper convex order bound

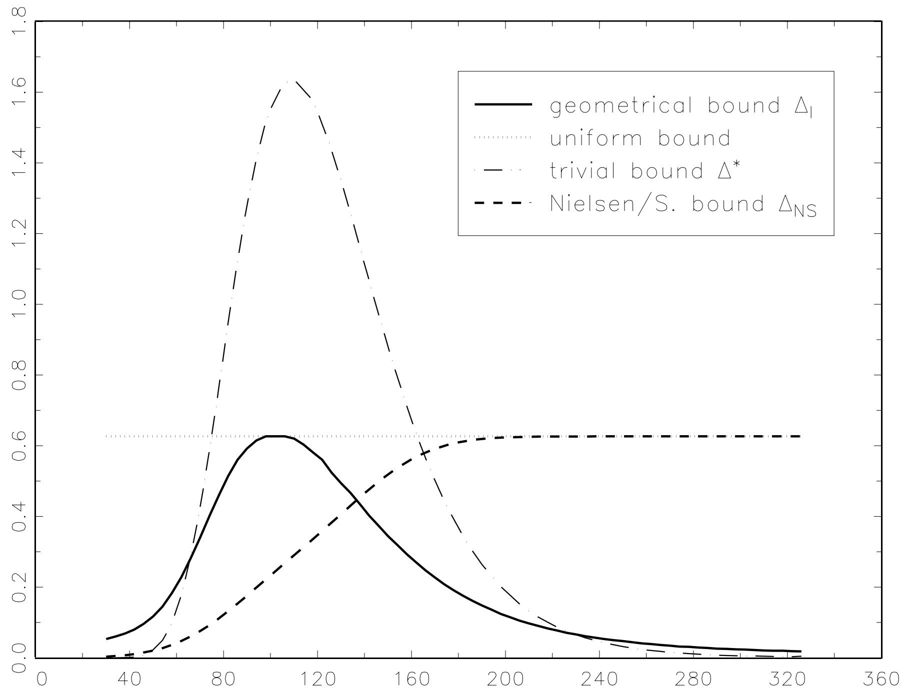

Figure 6: Upper bounds for the deviation of the expected payoff from its lower convex order bound

the variances is very small (1.57), which yields a good performance of Δ_l and Δ_{NS}. It can be seen, that the geometrical bound performs well for large values of t, whereas the Nielsen-Sandmann bound performs well for small values of t. So it might be a good idea to combine these two bounds to obtain a new one: $\min\{\Delta_l, \Delta_{NS}\}$.

We would like to compare our results to those of Nielsen and Sandmann (2003), who studied for exactly the data situation considered here several upper bounds for $E[(S-t)^+]-E[(S^l-t)^+]$. Among them the bound Δ_{NS} has been found to outperform any of the other bounds they considered – except for "far out-of-the-money options" (with strikes $t > 160$). Figure 6 shows, that our geometrical bound Δ_l performs better than Δ_{NS} already for strikes $t > 140$.

It has to be noted that Nielsen and Sandmann (2003) chose Z as the geometric average (GA) of the summands of S. But neither the lower bound $E[(S^l - t)^+]$ nor Δ_{NS} nor Δ_l differ significantly from their corresponding counterparts, when Z is choosen as throughout this article (i.e variance-maximizing (VM), cf. (10)). To give an impression of that, the following table compares lower and upper bounds *for the option price* for selected strikes t and either choices (VM) and (GA) of the conditioning random variable Z. $\delta = e^{-rT}$ denotes the discounting factor; the best lower and upper bounds are highlighted.

t	$\delta E[(S^l - t)^+]$		$\delta(E[(S^l - t)^+] + \Delta_{NS}(t))$		$\delta(E[(S^l - t)^+] + \Delta_l(t))$	
	VM	GA	VM	GA	VM	GA
50	50.0472	**50.0473**	50.06	**50.0488**	50.1511	50.1653
80	24.7443	**24.7461**	24.85	**24.8211**	25.1745	25.2198
90	17.9298	**17.9314**	18.08	**18.0553**	18.4550	18.5027
100	12.4754	**12.4759**	12.68	**12.6507**	13.0431	13.0898
110	**8.3864**	8.3857	8.643	**8.6138**	8.9364	8.9880
200	**0.1189**	0.1181	0.6719	0.6984	**0.2254**	0.2353

Table 1: Option price: Comparison of upper and lower bounds for two choices of the conditioning random variable Z

We don't want to conceal that Vanmaele et al. (2006), who compared the bounds from Nielsen and Sandmann (2003) with some other, found the upper bound 0.2081 for the price of the option with strike 200, which outperforms the bound 0.2254 from above slightly. Their bound is actually $\delta \cdot E[(S^u - 200)]$ (for a certain choice of Z, cf. (1) for the definition of S^u), whose computation involves a numerical integration, however.

4 Conclusions

4.1 Summary

By geometrical reasoning we derived bounds Δ_u resp. Δ_l for the deviation in terms of truncated first moments of a random variable S from an arbitrary upper resp. lower convex order bound of it. Any of the two bounds performs the better, the closer the corresponding convex order bound is to S in the convex order sense.

In order to compute the bounds no knowlegde of the distribution of S is required, except $\text{Var}(S)$.

We applied the results to the case, where S is a sum of lognormally distibuted random variables. As an example we considered to price Asian options. Based on the concept of convex order, lower and upper bounds for the expected payoff of such an option can be easily computed, cf. Dhaene et al. (2002a,b). We present bounds for their deviation from the true expected payoff, which are based on our geometrical approach. It turned out, that under certain circumstances they perform better than other bounds, e.g. the bound proposed by Nielsen and Sandmann (2003).

4.2 Benefits

The bounds Δ_u resp. Δ_l are especially helpful, if not both (upper and lower) convex order bounds are available at the same time.

On the one hand, in some situations one might find a conditioning Z (for example (10) in the lognormal setting), which leads to a lower bound S^l close to S in the convex order sense. However, it can easily happen that the corresponding S^l has no comonotonic structure, making its distribution (and thus $E[(S^l-t)^+]$) hardly explicitly computable. On the other hand, one might find a conditioning Z, such that the corresponding S^l has comonotonic structure and thus leads to an easy computation of the distribution of S^l. In fact, Deelstra et al. (2004) showed, that in the lognormal setting one can always find a Z, which leads to a comonotonic structure of S^l. But $E[(S^l-t)]$ could then be an almost arbitrarily bad lower bound. In these two cases one will benefit from a new method to derive a lower bound for $E[(S-t)^+]$, which is based on our results: If S^u is an upper convex order bound for S and $\Delta_u(t)$ an upper bound for $E[(S^u-t)^+]-E[(S-t)^+]$, then we get:

$$E[(S^u-t)^+]-\Delta_u(t) \leq E[(S-t)^+] \ . \tag{15}$$

If S^u is chosen as in (1) and has thus comonotonic structure (conditionally on Z), the computation of this new lower bound is always possible without exorbitant effort.

However, this computation requires a numerical integration. In order to avoid the latter, one might try to obtain an upper bound for $E[(S-t)^+]$, if the distribution of a lower convex order bound S^l of S is available, as follows:

$$E[(S-t)^+] \leq E[(S^l-t)^+]+\alpha(t) \ , \tag{16}$$

where α is an upper bound for $E[(S-t)^+]-E[(S^l-t)^+]$, e.g. Δ_{NS} (following Nielsen and Sandmann (2003)) or Δ_l (following our approach). In the numerical examples, which have been presented here, Δ_l turned out to outperform Δ_{NS} for large t.

4.3 Final remarks

The Nielsen-Sandmann approach yields the upper bound (16) (with $\alpha=\Delta_{NS}$), but it does not yield an upper bound for S in the convex order sense – the necessary condition $\lim_{t\to\infty}\Delta_{NS}(t)=0$ is not fulfilled. However, this approach might be promising if one is not primarily interested in an upper convex order bound for S and in its distribution, but merely in an upper bound for the truncated first moments of S, for example when pricing Asian or Basket options.

It is left for future research to check under which conditions the bounds (15) resp. (16) (with $\alpha = \Delta_l$) represent TFM-functions of a lower resp. upper convex order bound of S. In order to do that one has basically to check, whether the bounds are convex and decreasing functions having first derivative between -1 and 0. Recall that if a function is a TFM-function of a random variable, one gets the c.d.f. of the latter by taking the first derivative of the former one, cf. (T5).

Appendix: Proof of Lemma 1

Repeatedly we will utilize the fact, that a convex function with a positive (negative) right derivative is strictly increasing (strictly decreasing). For this and other properties of convex functions, cf. Roberts and Varberg (1973).

(i) q_t is convex, since it is a sum of convex functions. For its right derivative we obtain $q'_{t+}(s) = (F_X(s) - 1) + 1 = F_X(s) > 0$ for $s \in \{F_X > 0\}$, hence q_t must be strictly increasing on $\{F_X > 0\}$.

Obviously, $\{F_X > 0\}$ is either $\mathbb{R}$ or an interval of the form $[c, \infty)$. We have $\lim_{s\to\infty} q_t(s) = \infty$. On the other hand we have $F_X(c-) = 0$ and thus, using (T3), $g(c) = E(X) - c$. Substituting this into q_t we get $q_t(c) = (E(X) - c) - g(c)$, which can't be positive as a consequence of (T1). Using that q_t is strictly increasing, our argumentation leads to $q_t([c, \infty)) \supseteq (0, \infty)$. Similarly, $q_t(\mathbb{R}) \supseteq (0, \infty)$ can be shown.

(ii) g is a convex function, and we have $g'_+(s) = F_X(s) - 1 < 0$ for $s \in \{F_X < 1\}$. Hence g is strictly decreasing on $\{F_X < 1\}$. From (T4) we get immediately $\{F_X < 1\} = \{g > 0\}$, and hence $g(\{F_X < 1\}) = (0, \infty)$.

(iii) Consider the auxiliary function $v(z) = g^{-1}(g(t) + z)$, $z \in (0, \infty)$. v is as the inverse of a convex and strictly decreasing function also convex. Thus, a sufficient condition for v to be strictly decreasing is that $v'_+(z) < 0$ for all $z > 0$. This condition is indeed fulfilled: we have

$$v'_+(z) = 1/g'_+[g^{-1}(g(t) + z)] = 1/(F_X[g^{-1}(g(t) + z)] - 1) \ ,$$

and this is negative, since the value of F_X in the denominator is smaller than 1, which is due to the fact that the domain of g^{-1} is just $\{F_X < 1\}$.

Combining the latter with (i), the function $h_t(z) = z \cdot [q_t^{-1}(z) - v(z)]$, $z > 0$, turns out to be strictly increasing. □

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