On the relationship of local projection stabilization to other stabilized methods for one-dimensional advection-diffusion equations

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Abstract

We consider a singularly perturbed advection-diffusion two-point boundary value problem whose solution exhibits layers. Eliminating the enrichments in the one-level approach of the local projection stabilization we end up with the differentiated residual method (DRM) which coincide for piecewise linears with the streamline upwind Petrov-Galerkin (SUPG) method and for piecewise polynomials of degree \( r \geq 2 \) with the variational multiscale method (VMS). Furthermore, we show that in certain cases the stabilization parameter can be chosen in such a way that the piecewise linear part of the solution becomes nodal exact. In this way, we obtain explicite formulas for the stabilization parameter depending on the local meshsize, the polynomial degree \( r \) of the approximation space, and the data of the problem. We discuss the behaviour of different modes of the discrete solution for varying stabilization parameters.

Keywords: stabilized finite element method, singular perturbation, advection-diffusion equation

1 Introduction

We consider the two-point boundary value problem

\[-\varepsilon u'' + b(x)u' + c(x)u = f(x) \quad \text{in } (0,1), \quad u(0) = u(1) = 0, \quad (1) \{1.1\}\]
with a small positive parameter $0 < \varepsilon \ll 1$ and sufficiently smooth functions $b$, $c$, and $f$. We assume that

$$c(x) - \frac{1}{2} b'(x) \geq \gamma > 0, \quad x \in [0, 1],$$

which guarantees the unique solvability of the problem.

Standard Galerkin-type finite element methods exhibit spurious oscillations unless the mesh is very fine. Therefore, a number of stabilized methods (SUPG, Galerkin-least squares (GLS), residual free bubble (RFB), etc.) have been developed and extended both to the multi-dimensional case of advection-diffusion problems and to the incompressible Navier-Stokes equations; see [?] for the beginning and [?, ?] for a survey. Further, the variational multiscale method [?, ?] has been introduced as a framework for a better understanding of the fine-to-coarse scale effects and as a platform for the development of new numerical methods. Recently, stabilizations based on local projections become quite popular [?, ?], in particular due to its commutative properties in optimization problems and the stabilization properties equal to that of the SUPG [?].

For the constant coefficient case (with $c = 0$) and piecewise linear finite elements the close relations between the SUPG method, the residual free bubble approach, and the variational multiscale method are well-known. However, when using higher order finite elements the variational multiscale approach leads to a new stabilized method which seems to be not analyzed up to now. In order to distinguish this new method from the SUPG method we will call it variational multiscale (VMS) method.

The main objective of the paper is to analyze the VMS method and to give error estimates in several norms on different types of meshes. In Section 2 we shortly describe the variational multiscale approach leading to the new numerical method and show an alternative way for its derivation. Then, in Section ?? we give an error estimate in a mesh dependent norm which is related to the discrete bilinear form. It is important to note that in the case of higher order finite elements a new interpolation has been used to get the desired error estimates. Section ?? is devoted to $\varepsilon$-uniform error estimates on families of Shishkin meshes.

These estimates are based on a decomposition of the solution into a smooth and layer part, respectively, as well as a detailed study of approximation properties in and outside the layer region. It turns out that the error of the new VMS method to the interpolant is of order $k+1/2$ uniformly in $\varepsilon$ for piecewise polynomials of degree $k$. Using superconvergence properties in the case $k = 1$ the accuracy can be enhanced to almost second order. Finally, we show that by a proper post-processing the same bounds can be established for the error of the postprocessed numerical solution to the solution itself.

Notations. Throughout the paper $C$ will denote a generic positive constant that is independent of $\varepsilon$ and the mesh.

We use the standard Sobolev spaces $W^{k,p}(D)$, $H^k(D) = W^{k,2}(D)$, $H^k_0(D)$,
The weak formulation of (1) is given by:

\[ L^p(D) = W^{0,p}(D) \] for nonnegative integers \( k \) and \( 1 \leq p \leq \infty \) and write \((\cdot,\cdot)_D\) for the \( L^2(D) \) inner product. Here \( D \) is any measurable subset of \((0,1)\). Then, \( |\cdot|_{k,p,D} \) and \( \|\cdot\|_{k,p,D} \) are the usual Sobolev seminorm and norm on \( W^{k,p}(D) \). When \( D = (0,1) \) we drop \( D \) from the notation for simplicity. We will also simplify the notation in the case \( p = 2 \) by setting \( \|\cdot\|_{k,D} = \|\cdot\|_{k,2,D} \) and \( |\cdot|_{k,D} = |\cdot|_{k,2,D} \).

## 2 One-level local projection stabilization

The weak formulation of (1) is given by:

Find \( u \in V := H^1_0(0,1) \) such that for all \( v \in V \)

\[
a(u,v) := \varepsilon(u',v') + (bu' + cu,v) = (f,v).
\] (3) \{2.1\}

Let \( 0 = x_0 < x_1 < \cdots < x_N = 1 \) be a partition \( T_h \) of \([0,1]\) into cells \( K \in T_h \). The local projection stabilization is based on an approximation space \( V_h \) and a projection space \( D_h \). In the one-level variant both spaces live on the same mesh \( T_h \). As shown, in a more general setting, in [?], the keypoint in the error analysis of the local projection stabilization is the existence of an interpolation in \( V_h \) such that the interpolation error is orthogonal to \( D_h \). This can guaranteed if the approximation space \( V_h \) is rich enough compared to \( D_h \). For some \( r \in \mathbb{N} \), let the solution and projection spaces are defined by

\[
V_h := \{v_h \in H^1_0(0,1) : v_h|_K \in P^{r+}_r(K) \quad \forall K \in T_h\},
\]

\[
D_h := \{q_h \in L^2(0,1) : q_h|_K \in P^{r-1}_r(K) \quad \forall K \in T_h\}
\]

where \( P_m, m = 0,1, \ldots, \) denotes the space of polynomials of degree less than or equal to \( m \), and \( P^+_r = P_r + \text{span}(x^{r+1}) = P_{r+1} \) is the enriched \( P_r \) space.

In the one-dimensional case, we have an explicit characterization of the above mentioned interpolation.

**Theorem 2.1** There is an interpolation operator \( j_h : H^1_0(0,1) \to V_h \) such that

\[
(j_h w - w, q_h) = 0 \quad \forall q_h \in D_h, \ w \in H^1_0(0,1) \quad (4) \{5\}
\]

\[
|j_h w - w|_{m,K} \leq C h^{r+1-m}_K \|w\|_{r+1,K} \quad \forall w \in H^{r+1}(K), \ K \in T_h. \quad (5) \{6\}
\]

First we note that counting the number of conditions, we see that uniqueness of such an interpolation can not be expected. On a macro cell, a continuous, piecewise polynomial interpolation of degree \( r \) has \( 2r+1 \) degrees of freedom. To guarantee that \( j_h w \) is continuous, the nodal functionals \( w \mapsto w(x_{2i}) \) and \( w \mapsto w(x_{2i+2}) \) should be used. To gether with the \( r \) orthogonality conditions (4), we have \( 2r + 1 - 2 - r = r - 1 \geq 0 \) remaining degrees of freedom. In the case of piecewise linears, \( r = 1 \), we have no freedom in chosing \( j_h w \), however for higher order approximations \( r \geq 2 \) there is no unique choice of the interpolant.
Now, we give explicit examples where we will distinguish the cases \( r \) odd and \( r \) even. We start defining local interpolants on the reference macro \((-1, +1)\). Let \( r \) be an odd number. Then, consider the nodal functionals for \( i = 0, \ldots, r - 2 \)

\[
N_{2i}(v) = \int_{-1}^{0} v(\xi)L_{i}(2\xi + 1) \, d\xi \quad N_{2i+1}(v) = \int_{0}^{+1} v(\xi)L_{i}(2\xi - 1) \, d\xi,
\]

where \( L_{i}, i = 0, 1, \ldots, \) denote the the Legendre polynoms of degree \( i \) on \((-1, +1)\) normalized such that \( L_{i}(1) = 1, \, i = 0, 1, \ldots \). Further, let

\[
N_{2r-2}(v) = v(-1) \quad N_{2r-1}(v) = v(+1) \quad N_{2r}(v) = \int_{-1}^{+1} v(x)L_{r-1}(x) \, dx.
\]

The above set of nodal functionals determines uniquely a continuous, piecewise polynomial of degree \( r-1 \). Indeed, setting \( N_{j}(v) = 0, \, j = 0, 1, \ldots, 2r \), we get from the \( L^{2}(-1, +1) \) orthogonality of the Legendre polynomials the representation

\[
v(\xi) = \begin{cases} 
AL_{r}(2\xi + 1) + BL_{r-1}(2\xi + 1) & \text{for } \xi \in (-1, 0) \\
CL_{r}(2\xi - 1) + DL_{r-1}(2\xi - 1) & \text{for } \xi \in (0, 1)
\end{cases}
\]

with unknown coefficients \( A, B, C, \) and \( D \). \( N_{2r-2}(v) = N_{2r-1}(v) = 0 \) and continuity of \( v \) at \( \xi = 0 \) result into \( (r \text{ odd}) \)

\[
A - B = 0, \quad C + D = 0, \quad A + B = -C + D.
\]

On each \( K = (x_{i}, x_{i+1}) \) a local interpolant \( j_{h}^{K}w \in P_{r}(K) \) is uniquely defined by the \( r + 1 \) conditions

\[
\begin{align*}
    j_{h}^{K}w(x_{i}) &= w(x_{i}) \quad j_{h}^{K}w(x_{i+1}) = w(x_{i+1}) \quad (j_{h}^{K}w - w, q)_{K} = 0 \quad \forall q \in P_{r-1}(K).
\end{align*}
\]

The global interpolant \( j_{h}w \) defined by

\[
 j_{h}w|_{K} = j_{h}^{K}(w|_{K}) \quad \forall K \in T_{h}
\]

belongs to \( V_{h} \) by construction. Since \( j_{h}^{K}w = w \) for all \( w \in P_{r}(K) \), we obtain (5) by means of the Bramble-Hilbert-Lemma. Taking into consideration that \( q_{2h}|_{K} \in P_{r-1}(K) \), we have \( (j_{h}^{K}w - w, q_{2h})_{K} = 0 \) for all \( K \in T_{h} \) from which (4) follows by summation. \( \square \)

Let \( \pi_{2h} : L^{2}(0, 1) \to D_{2h} \) denote the \( L^{2} \) projection, \( \text{id} : L^{2}(0, 1) \to L^{2}(0, 1) \) the identity, and \( \kappa_{2h} := \text{id} - \pi_{2h} \) the fluctuation operator. Finally, we define the stabilizing term

\[
S_{h}(u_{h}, v_{h}) := \sum_{M \in T_{2h}} \alpha_{M} \langle \kappa_{2h}u'_{h}, \kappa_{2h}v'_{h} \rangle_{M}
\]
and the mesh-dependent norm

$$|||v||| := \left( \varepsilon |v|^2 + \gamma \|v\|^2 + \sum_{M \in T_h} \alpha_M \|\kappa_{2h} v'\|_{0,M}^2 \right)^{1/2}$$

Our stabilized method reads

$$a(u_h, v_h) + S_h(u_h, v_h) = (f, v_h). \quad \text{(6) \{LPS\} \{theo1\}}$$

**Theorem 2.2** Let $u$ and $u_h$ be the solutions of the weak formulation (3) and the stabilized method (6), respectively. Assume that $u \in H^2_0(0,1) \cap \mathcal{H}^{r+1}(0,1)$ and chose $\alpha_M \sim h_M$. Then, the error estimate

$$|||u - u_h||| \leq C (\varepsilon^{1/2} + h^{1/2}) h^r \|u\|_{r+1}$$

holds.

**Proof.** Using integration by parts, the bilinear form $a$ satisfies

$$a(v_h, v_h) = \varepsilon |v_h|^2 + (c - \frac{1}{2} b', v_h^2) \geq \varepsilon |v_h|^2 + \gamma \|v_h\|^2$$

and we have

$$a(v_h, v_h) + S_h(v_h, v_h) \geq |||v_h|||^2 \quad \forall v_h \in V_h.$$ 

Setting $w_h := j_h u - u_h \in V_h$ we have from (3) and (6)

$$|||j_h u - u_h|||^2 \leq a(j_h u - u, w_h) + S_h(j_h u - u, w_h) + S_h(u, w_h). \quad \text{(7) \{99\}}$$

We use integration by parts for the advection term

$$(b(j_h u - u)' , w_h) = -(b(j_h u - u), w_h') - (b'(j_h u - u), w_h)$$

and estimate each term on the right hand side of (7) individually

$$|\varepsilon((j_h u - u)' , w_h')| \leq C \varepsilon^{1/2} h^r \|u\|_{r+1} \varepsilon^{1/2} \|w_h\|_1 \leq C \varepsilon^{1/2} h^r \|u\|_{r+1} \|w_h\||$$

$$|(c - b')(j_h u - u), w_h)| \leq C h^{r+1} \|u\|_{r+1} \|w_h\|_0 \leq C h^{r+1} \|u\|_{r+1} \|w_h\||$$

$$|S_h(j_h u - u, w_h)| \left( \sum_{M \in M_h} \alpha_M \|\kappa_{2h}(j_h u - u)'\|_{0,M}^2 \right)^{1/2} \leq C h^{r+1/2} \|u'\|_r \|w_h\||$$

$$|S_h(u, w_h)| \leq \left( \sum_{M \in M_h} \alpha_M \|u' - \pi_{2h} u'\|_{0,M}^2 \right)^{1/2} \leq C h^{r+1/2} \|u'\|_r \|w_h\||.$$
It remains to estimate the term \((b(j_h u - u), w'_h)\). For this let \(\overline{b}\) be the piecewise constant approximation of \(b\) with respect to \(T_{2h}\). Then,

\[
(b(j_h u - u), w'_h) = ((b - \overline{b})(j_h u - u), w'_h) + (\overline{b}(j_h u - u), w'_h)
\]

\[
|b(j_h u - u), w'_h)| \leq Ch\|b\|_{1,\infty} h^{r+1}\|u\|_{r+1}\|w_h\|_1
\]

\[
\leq Ch^{r+1}\|u\|_{r+1}\|w_h\|_0 \leq Ch^{r+1}\|u\|_{r+1}\|w_h\|_1
\]

\[
\square
\]

3 Relationship of the one-level local projection stabilization to the differentiated residual method

We consider in the following the one-level approach of the local projection stabilization for the pair \((V^+_h, D_h) = (P^+_r, P^{\text{disc}}_{r-1})\) in the special case that \(b = \text{const}\), \(c = 0\), and \(f\) piecewise \(P^{r-1}\). In the one-dimensional case we have \(P^+_r = P^{r+1}\), which means that the enriched approximation space is the next higher polynomial space. We split the approximation space into the direct sum

\[
V^+_h = V_h \oplus B_h, \quad B_h = \bigoplus_{K \in T_h} \text{span} \, b_K,
\]

where \(b_K \in H^1_0(K)\) for \(K = (x_i, x_{i+1})\) is given by

\[
b_K(x) = L_{r+1} \left( \frac{2x - x_i - x_{i+1}}{x_{i+1} - x_i} \right) - L_{r-1} \left( \frac{2x - x_i - x_{i+1}}{x_{i+1} - x_i} \right)
\]

and extended by zero on \(\Omega \setminus K\). Let us split the solution \(u^+_h \in V^+_h\) of the local projection stabilization in the same way

\[
u^+_h = u_h + \sum_{K \in T_h} u_K b_K.
\]

Taking into consideration that for \(v_h \in V_h\) we have \(v'_h|_K \in P^{r-1}(K)\), the \(L_2\) projection becomes \(\pi_h v'_h = v'_h\) thus the fluctuation \(\kappa_h v'_h\) vanishes. We get

\[
\varepsilon(u'_h, v'_h) + (bu'_h, v_h) + \sum_{K \in T_h} u_K (\varepsilon(b'_K, v'_h) + (b b'_K, v_h)) = (f, v_h) \quad \forall v_h \in V_h
\]

\[
\varepsilon(u'_h, b'_K) + (bu'_h, b_K) + u_K [\varepsilon(b'_K, b'_K) + (b b'_K, b_K) + \tau_K(\kappa_h b'_K, \kappa_h b'_K)] = (f, b_K) \quad \forall K \in T_h
\]

On the choice of the LPS-parameter \(\tau_K\) Model problem

\[-\varepsilon u'' + bu' + cu = f \quad \text{in } (0,1), \quad u(0) = u(1) = 0\]
\( (V_h^+, D_h) = (P_r^+, P_{r-1}^{\text{disc}}) = (P_{r+1}, P_{r-1}^{\text{disc}}) \)

Stabilized method Find \( u_h^+ \in V_h^+ \) such that for all \( v_h^+ \in V_h^+ \)

\[
\varepsilon (u_h^+, v_h^+) + (b(u_h^+)' + cu_h^+ + v_h^+ \bigr) + \sum_{K \in T_h} \tau_K (\kappa_h(b(u_h^+)'), \kappa_h(b(v_h^+)')) = (f, v_h^+)
\]

Splitting

\[
u_h^+ = u_h + \sum_{K \in T_h} u_K (L_r^{K+1} - L_r^{K-1})
\]

Elimination of enrichments I Set \( K = (x_i, x_{i+1}) \), \( h_K = x_{i+1} - x_i \) and split the space

\[
V_h^+ = V_h \oplus B_h, \quad B_h = \text{span} \bigoplus_{K \in T_h} \varphi_{r,K}
\]

where the bubble space is spanned by

\[
\varphi_{r,K}(x) = \begin{cases} L_{r+1} \left( \frac{2x - x_i - x_{i+1}}{h_K} \right) - L_{r-1} \left( \frac{2x - x_i - x_{i+1}}{h_K} \right) & \text{for } x \in K \\ 0 & \text{for } x \not\in K \end{cases}
\]

Important properties

\[
(v_h', \varphi_{r,K}') = 0 \quad \forall v_h \in V_h, \quad \forall K \in T_h
\]

\[
\pi_h v_h' = v_h' \quad \forall v_h \in V_h \iff \kappa_h v_h' = 0 \quad \forall v_h \in V_h
\]

Elimination of enrichments II Case \( b = \text{const}, \ c = 0 \), and \( f \) piecewise \( P_{r-1} \), using

\[
(b \varphi_{r,K}, \varphi_{r,K}) = 0, \quad \pi_h \varphi_{r,K} = 0, \quad \varphi_{r,K} = \psi^{(r-1)}_K, \quad \Psi^{(j)}_K = 0 \text{ on } \partial K
\]

for \( j = 0, 1, \ldots, r - 1 \) we get

\[
\varepsilon (u_h', v_h') + (bu_h', v_h') + \sum_{K \in T_h} u_K (b \varphi_{r,K}', v_h) = (f, v_h) \quad \forall v_h \in V_h
\]

\[
u_K \left\{ \varepsilon (\varphi_{r,K}', \varphi_{r,K}') + \tau_K b^2 (\varphi_{r,K}', \varphi_{r,K}') \right\} = (f - bu_h', \varphi_{r,K}) \quad \forall K \in T_h
\]

\[
u_K = (f - bu_h')^{(r-1)} |(K^{(-1)}(1, \psi_K)) / (\varepsilon + \tau_K b^2) |\varphi_{r,K}|^2_{1,K}
\]
Differentiated residual method DRM – SUPG (r=1)

Find \( u_h \in V_h \) such that for all \( v_h \in V_h \)

\[
\varepsilon(u_h', v_h') + (bu_h + cu_h, v_h) + \sum_{K \in T_h} \gamma_K((bu_h^{(r-1)}, (bv_h^{(r-1)})_K
= (f, v_h) + \sum_{K \in T_h} \gamma_K(f^{(r-1)}, (bv_h^{(r-1)})_K
\]

\[
\gamma_K = \frac{(1, \psi_K)^2}{(\varepsilon + \tau_K b^2) h_K |\varphi_{r,K}|_{1,K}^2}
\]

VMS Hughes/Sangalli 05/07

Error bound on uniform and layer adapted meshes Tob 06

Relationship LPS and DRM Relationship of LPS to DRM

Theorem 3.1 (Tob 2008) Assume \( b = \text{const.} \), \( c = 0 \), and \( f \) piecewise \( P_{r-1} \).
Eliminating the enrichment in the \( (P_r^+, P_{r-1}^{\text{disc}})\)-LPS gives the \( P_r\)-DRM with the correct scaling in both the convection dominated and diffusion dominated limit.

Recursion formula for \( P_{r+1}\)-DRM

Theorem 3.2 (Tob 2008) Assume \( b = \text{const.} \), \( c = 0 \), and \( f \) piecewise \( P_{r-1} \).
Eliminating the highest order mode in the \( P_{r+1}\)-DRM results in the \( P_r\)-DRM.
The \( P_1\)-DRM is equal to the SUPG.

Recursion formula Recursion formula for \( P_{r+1}\)-DRM I Assume \( b = \text{const.} \), \( c = 0 \), and \( f \) piecewise \( P_{r-1} \). Set

\[
V_r = \{ v_h \in H^1_0(0,1) : \varphi_h| \in P_r(K), K \in T_h \}, \quad r \geq 1
\]

and use the splitting

\[
V_{r+1} = V_r \oplus \text{span} \bigoplus_{K \in T_h} \varphi_{r,K}.
\]

\[
\varepsilon(u_r', v_r') + (bu_r', v_r) + \sum_{K \in T_h} u_K(b\varphi_{r,K}', v_r) = (f, v_r) \quad \forall v_r \in V_r
\]

\[
u_K \left\{ \varepsilon|\varphi_{r,K}|^2_1 + \gamma_{r+1} b^2 |\varphi_{r,K}|_{r+1}^2 \right\} = (f - bu_r', \varphi_{r,K}) \quad \forall K \in T_h
\]

Recursion formula for \( P_{r+1}\)-DRM II Integrating by parts

\[
(f - bu_r', \varphi_{r,K}) = (f - bu_r', \psi_K^{(r-1)}) = (-1)^{r-1}(f - bu_r')^{(r-1)}|_K (1, \psi_K)
\]

\[
(b\varphi_{r,K}', v_r) = -(bu_r', \psi_K^{(r-1)}) = (-1)^r(bu_r')^{(r-1)}|_K (1, \psi_K)
\]
\( P_r\)-DRM

\[
\varepsilon (u'_r, v'_r) + (bu'_r, v_r) + \sum_{K \in T_h} \gamma_r ((bu'_r)^{(r-1)}, (bv'_r)^{(r-1)})_K \\
= (f, v_r) + \sum_{K \in T_h} \gamma_r (f^{(r-1)}, (bu'_r)^{(r-1)})_K
\]

Recursion formula for \( P_{r+1}\)-DRM III

**Theorem 3.3** Assume \( b = \text{const} \), \( c = 0 \), and \( f \) piecewise \( P_{r-1} \). Eliminating the highest order mode in the \( P_{r+1}\)-DRM results in the \( P_r\)-DRM. The \( P_1\)-DRM is equal to the SUPG.

\[
\gamma_r = \frac{(1, \psi_K)^2}{\{\varepsilon |\varphi_{r,K}|^2 + \gamma_{r+1} b^2 |\varphi_{r+1,K}|^2\} h_K}
\]

DRM stabilization parameters, \( r \geq 1 \) Optimal SUPG parameter (nodal exact solution in the constant coefficient case) leads to

DRM stabilization parameters, \( r \geq 1 \)

\[
\gamma_r = \frac{h_{K,2}^{2r-1}}{\alpha_r b} \Phi_r(q_K), \quad q_K = \frac{b_K h_K}{2\varepsilon}, \quad \alpha_r = \frac{2[(2r-1)!]^2}{[(r-1)!]^2}
\]

\[
\Phi_{r+1}(q) = \frac{1}{\Phi_r(q)} - \frac{2r + 1}{q}, \quad \Phi_1(q) = \coth q - \frac{1}{q}. \]

Observations for the linear part

- Theory \( \tau_K \sim \tau_0 h_K \) but \( \tau_0 \) ???
- \( P^+_1 \) discretization
  - \( \tau_0 \gg 1 \) oscillations
  - \( \tau_0 \to 0 \) smearing
- \( P^+_2 \) discretization
  - \( \tau_0 \gg 1 \) smearing
  - \( \tau_0 \to 0 \) oscillations

Weighting functions

\[
\Phi_1(q) = \coth q - \frac{1}{q}, \quad \Phi_{r+1}(q) = \frac{1}{\Phi_r} - \frac{2r + 1}{q}, \quad r = 1, 2, \ldots
\]
\[
\lim_{q \to +0} \Phi_r(q) = 0, \quad \lim_{q \to \infty} \Phi_r(q) = 1, \quad r = 1, 2, \ldots
\]

- How behave the different modes for increasing $\gamma_K$?

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