Fakultät für Mathematik, Institut für Mathematische Stochastik
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Tutorial Stochastic Processes, (Serie 5)

21.) Prove with Extension I of the Ito Lemma, that

$$a) \qquad X_t = (W_t + t)\, e^{-W_t - t/2}, \qquad b) \qquad Y_t = e^{t/2} \cos W_t$$

$$c) \quad Z_t = W_t^4 - 6\, t\, (W_t^2 - t) - 3\, t^2$$

are martingales with respect to the filtration, generated by $(W_t)_{t \geq 0}$.
Hint: Prove that the processes X_t, Y_t and Z_t are Ito integrals!

Since Z_t in c) is a martingale and $Z_t = W_t^4 + A_t$ with $A_t = -6\, t\, (W_t^2 - t) - 3\, t^2$ we find the stochastic process A_t in **Problem 20***.

22.) Let W_t be a standard Wiener process and η_t a bounded and adapted to W_t process, $0 \leq t \leq T < \infty$. The process ξ_t is defined by the integral equation

$$\xi_t = \int_0^t \eta_u dW_u - \frac{1}{2} \int_0^t (\eta_u)^2 du\,.$$

a) Put $Z_t = \exp(\xi_t)$, $0 \leq t \leq T < \infty$. Prove that Z_t satisfies the stochastic integral equation $Z_t = 1 + \int_0^t Z_u \eta_u dW_u$.

b) Put $Y_t = 1/Z_t$, $0 \leq t \leq T < \infty$. Prove that Y_t satisfies the stochastic differential equation $dY_t = Y_t(\eta_t)^2 dt - Y_t \eta_t dW_t$.
Hint: Use the Extension II of the Ito Lemma.

23.) Let B_t and W_t be two independent standard Wiener processes on $[0\,, T]$.
a.) Show that $X_t = (B_t + W_t)/\sqrt{2}$ is also a standard Wiener process and calculate the correlation coefficient between W_t and X_t.
b.) Show the product rule

$$d(W_t\, B_t) = W_t\, dB_t + B_t\, dW_t\,.$$

Note that there is no *correction term dt* since W_t and B_t are independent. In case of $P(W_t = B_t) = 1$ we obtained $d(W_t^2) = 2W_t dW_t + dt$.

24.) Let W_t be a standard Wiener process and $\alpha > 0$ a constant. Define $X_t = e^{-\alpha t}\, W(e^{2\alpha\, t})$. Find its mean and covariance function. Show, that X_t is a stationary process.

25.) Let W_t be a standard Wiener process. Find dX_t for

a) $X_t = t \exp\{W_t\}$, b) $X_t = 2 + t + W_t^2$.

Examination Stochastic Processes

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Please note the following:

- The exam consists of 6 problems. You do not have to solve the individual problems completely, partial solutions are also possible. You should clearly display your approach and way to solution.
- You can reach a maximum of **40 points**. For passing the exam a total of **13 points** is sufficient.
- You are allowed to use: Pocket calculators, text books, mathematical and/or statistical tables, manuscripts and notes from lectures and/or exercises.

1.) (10 points) Suppose $X_1 \sim N(0,9)$, $X_2 \sim N(3,1)$, $X_3 \sim N(1,4)$ and let X_1, X_2, X_3 be independent. Define $Y_1 = 2\,X_2 - 3\,X_1 - 2$ and $Y_2 = 3\,X_3 - X_2$.

a) Find EY_2 and $Var(Y_1)$!

b) Calculate the covariance-matrix $\Sigma_{(Y_1,Y_2)} = \Big(E[(Y_i - EY_i)(Y_j - EY_j)]\Big)_{i,j=1,2}$!

c) Calculate the correlation coefficient $\rho(Y_1, Y_2)$!

d) Find $E\Big((Y_2)^2 \,\Big|\, \sigma(X_1, X_2)\Big)$!

2.) (6 points) Let N_t, $t \geq 0$ is a homogeneous Poisson process with intensity $\lambda > 0$. Suppose $t_1 < t < t_2$. Calculate $p = P(N_{t_1} = 1\,,\, N_t = 3 \,|\, N_{t_2} = 4)$!

3.) (6 points) Let W_t, $t \geq 0$ be a standard Wiener process ($\sigma^2 = 1$). Define $X_t = 1 + W_t^2$, $t \geq 0$. Calculate $Cov(X_s, X_t)$ for $s < t$. **Hint:** Note $EW_u^4 = 3\,u^2$ for $u > 0$.

4.) (5 points) Let W_t, $t \geq 0$ be a standard Wiener process ($\sigma^2 = 1$). Define $Y_t = W_t\, e^{2\,W_t - 2\,t}$. Find dY_t !
Hint: Use Extension I of the Ito Lemma.

5.) (5 points) Put $Y_t = t + \ln(X_t) = f(t, X_t)$, $t \geq 0$, where

$$X_t = X_0 + \int_0^t (2\,s^2 - 1)\,X_s\,ds + \int_0^t 2\,s\,X_s\,dW_s \quad \text{with} \quad X_0 = 1\,,$$

where W_t, $t \geq 0$, is the standard Wiener process ($\sigma^2 = 1$). Find with Extension II of the Ito Lemma, that the process Y_t is a martingale with respect to the filtration, generated by $(W_t)_{t\geq 0}$.
Hint: Prove that the process Y_t is a Ito integral!

6.) (8 points) The following statements are TRUE or FALSE. So just answer TRUE, since ... or FALSE, since .... giving only a short explanation.

a) The random variables X_3 and Y_1 in **Problem 1** are uncorrelated.

b) Let $N(t)$, $t \geq 0$ is a homogenuous Poisson process with intensity $\lambda > 0$ and $t_1 < t < t_2$ (see **Problem 2**). Then $q = P(N(t_1) = 2\,,\, N(t) = 2 \,|\, N(t_2) = 5) = 0\,.$

c) The process X_t in **Problem 3** is a stationary process!

d) Let $N_t, t \geq 0$ be a Poisson process with intensity 2. The process $X_t = N_t - 4\,t$ is a martingale with respect to its natural filtration $\mathcal{F}_t = \mathcal{F}(N(u),\ 0 \leq u \leq t)$!