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Tutorial Stochastic Processes
(Serie 2, 2011/12)

6. Example 2.3: Let $X_1, X_2, ...$ be independent identically distributed (i.i.d.) random variables and $S_0 = 0, S_k = X_1 + ... + X_k$, $k \geq 1$, $k = 1, 2, .., n$. The sequence $\{S_n\}_{n\geq 0}$ is called a random walk. Prove that the random process $\{S_n\}_{n\geq 0}$ has stationary and independent increments.

7. In case of the random walk $\{S_n\}_{n\geq 0}$ in Ex.2.1 with $P(X_k = \pm 1) = 1/2$ find the expectation vector $m(k) = E(S_k)$, $k = 0, 1, ..., n$,
   the covariance function $C_S(k, m) = Cov(S_k, S_m)$, $k, m \in \{1, 2, ...n\}$
   and the covariance matrix $\Sigma_S = \Big(Cov(S_k, S_m)\Big)_{m,k=1,2,...,n}$ of the vector $S = (S_1, S_2, ..., S_n)$!

8. In Ex 1.8 the two-dimensional Gaussian density was defined as

$$f_{(X_1,X_2)}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

$$\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\Big(\frac{x_1-\mu_1}{\sigma_1}\Big)^2 - 2\rho\frac{(x_1-\mu_1)}{\sigma_1}\frac{(x_2-\mu_2)}{\sigma_2} + \Big(\frac{x_2-\mu_2}{\sigma_2}\Big)^2\right]\right\} .$$

   Show that it coincides with formula (2.1) of the n-dimensional Gaussian density in case $n = 2$.

9. Let N_T be a homogeneous Poisson process with parameter $\lambda > 0$. Calculate the expectation function $m_N(t) = E\,N(t)$, the covariance-function $Cov_N(s, t)$ and $E(N(t)\,N(s))$.

10. Let N_T be a homogeneous Poisson process with parameter $\lambda > 0$. Calculate for $t_1 < t_2$
    a) $P(N(t_2) = k_2 \,|\, N(t_1) = k_1)$,
    b) $P(N(t_1) = k_1 \,|\, N(t_2) = k_2)$.

11. Let $\{X(t);\, t \geq 0\}$ and $\{Y(t);\, t \geq 0\}$ be independent homogeneous Poisson processes with parameters λ_1 and λ_2, respectively. Define
    a) $Z_1(t) = X(t) + Y(t)$,
    b) $Z_2(t) = X(t) - Y(t)$ and
    c) $Z_3(t) = X(t) + k$ (Here $k \geq 0$ is a fixed integer).
    Which of the processes are such with independent increments, which of them are Poisson processes?