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Tutorial Stochastic Processes, (Serie 5 2013/14)

21.) Prove with Extension I of the Ito Lemma, that

$$a) \qquad X_t = (W_t + t)\, e^{-W_t - t/2}, \qquad b) \qquad Y_t = e^{t/2} \cos W_t$$

$$c) \quad Z_t = W_t^4 - 6\,t\,(W_t^2 - t) - 3\,t^2$$

are martingales with respect to the filtration, generated by $(W_t)_{t \geq 0}$.
Hint: Prove that the processes X_t, Y_t and Z_t are Ito integrals!

Since Z_t in c) is a martingale and $Z_t = W_t^4 + A_t$ with $A_t = -\,6\,t\,(W_t^2 - t) - 3\,t^2$ we find the stochastic process A_t in **Problem 20***.

22.) Let W_t be a standard Wiener process and η_t a bounded and adapted to W_t process, $0 \leq t \leq T < \infty$. The process ξ_t is defined by the integral equation

$$\xi_t = \int_0^t \eta_u dW_u - \frac{1}{2} \int_0^t (\eta_u)^2 du\,.$$

a) Put $Z_t = \exp(\xi_t)$, $0 \leq t \leq T < \infty$. Prove that Z_t satisfies the stochastic integral equation $Z_t = 1 + \int_0^t Z_u \eta_u dW_u$.

b) Put $Y_t = 1/Z_t$, $0 \leq t \leq T < \infty$. Prove that Y_t satisfies the stochastic differential equation $dY_t = Y_t(\eta_t)^2 dt - Y_t \eta_t dW_t$.
Hint: Use the Extension II of the Ito Lemma.

23.) Let B_t and W_t be two independent standard Wiener processes on $[0\,, T]$.
a.) Show that $X_t = (B_t + W_t)/\sqrt{2}$ is also a standard Wiener process and calculate the correlation coefficient between W_t and X_t.
b.) Show the product rule

$$d(W_t\, B_t) = W_t\, dB_t + B_t\, dW_t\,.$$

Note that there is no *correction term dt* since W_t and B_t are independent. In case of $P(W_t = B_t) = 1$ we obtained $d(W_t^2) = 2W_t dW_t + dt$.

24.) Let W_t be a standard Wiener process and $\alpha > 0$ a constant. Define $X_t = e^{-\alpha t}\, W(e^{2\alpha\, t})$. Find its mean and covariance function. Show, that X_t is a stationary process.

25.) Let W_t be a standard Wiener process. Find dX_t for

a) $X_t = t \exp\{W_t\}$, b) $X_t = 2 + t + W_t^2$.