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Tutorial Stochastic Processes, (Serie 4, 2013/14)

Problems with * are additional problems.

16.) Suppose $\Omega = \{1,2,3,4,5,6\}$ with $P(\{i\}) = 1/6$ for $i = 1,2,3,4,5,6$. Define $X(\omega) = \omega = \{\text{face value on die}\}$ with $P(X = i) = 1/6$ for $i = 1,2,3,4,5,6$
and $Y(\omega) = \begin{cases} 2 & \text{if} \quad \omega \in \{1,3,6\} \\ 3 & \text{if} \quad \omega \in \{2,5\} \\ 5 & \text{if} \quad \omega = 4 \end{cases}$. Find $Z = E(X|Y)$!

17.) a) Let X be uniform on $(0\,,\,1]$, $C_1 = (0\,,\,1/2]$, $C_2 = (1/2\,,\,1]$, and $\mathcal{F}_2 = \sigma(C_1, C_2)$. Find $V = E(X|\mathcal{F}_2)$!
b) Remember Example 3.2 with $\mathcal{F}_4 = \sigma(A_1, A_2, A_3, A_4)$.
Here $C_1 = A_1 \cup A_2, C_2 = A_3 \cup A_4$
and $Z(\omega) = E(X|\mathcal{F}_4) = \begin{cases} 1/8 & \text{if} \quad \omega \in A_1 \\ 3/8 & \text{if} \quad \omega \in A_2 \\ 5/8 & \text{if} \quad \omega \in A_3 \\ 7/8 & \text{if} \quad \omega \in A_4 \end{cases}$. Find $U = E(Z|\mathcal{F}_2)$!

18.) Let $(W(t),\ t \geq 0)$ be a standard Wiener Process and $(N(t),\ t \geq 0)$ a homogenuos Poisson Process mit dem Parameter $\lambda > 0$. Prove that

(a) $W^2(t) + 2\,W(t) - t$, (d) $N(t) - \lambda t$,
(b) $W^3(t) - 3\,t\,W(t)$, and (e*) $(N(t) - \lambda t)^2 - \lambda t$,
(c*) $\exp\{-\theta W(t) - \frac{1}{2}\theta^2\, t\}$ (f*) $\exp\{-\theta N(t) + \lambda t(1 - e^{-\theta})\}$

are martingales with respect to its natural filtration

$\mathcal{F}_t = \mathcal{F}(W(u),\ 0 \leq u \leq t)$ or $\mathcal{F}_t = \mathcal{F}(N(u),\ 0 \leq u \leq t)$.

(g) Show that $W^3(t)$ and $N(t)$ do not be martingales.

19.) Let $(W(t),\ t \geq 0)$ be a standard Wiener Process. Calculate
(4.1) $E \int_0^t W_s\, dW_s = 0$ and
(4.2) $E(\int_0^T W_s\, dW_s)^2 = \int_0^T E(W_s^2)\, ds$

20*.) Try to find a stochastic process A_t such that $(W_t^4 + A_t\,,\,\mathcal{F}_t)$ is a martingale with respect to the filtration $\mathcal{F}_s = \sigma(W_u, 0 \leq u \leq s)$, $s \geq 0$.
Hint: First calculate $E\Big([(W_t - W_s) + W_s]^4 \,|\, \mathcal{F}_s\Big)$ for $s < t$.