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Problems from Examinations Stochastic Processes 2010 and 2013

- You do not have to solve the individual problems completely, partial solutions are also possible. You should clearly display your approach and way to solution.
- You are allowed to use: Pocket calculators, text books, mathematical and/or statistical tables, manuscripts and notes from lectures and/or exercises.

1.) (10 points) Suppose $X_1 \sim N(0,9)$, $X_2 \sim N(3,1)$, $X_3 \sim N(1,4)$ and let X_1, X_2, X_3 be independent. Define $Y_1 = 2\,X_2 - 3\,X_1 - 2$ and $Y_2 = 3\,X_3 - X_2$.

a) Find EY_2 and $Var(Y_1)$!

b) Calculate the covariance-matrix $\Sigma_{(Y_1,Y_2)} = \Big(E[(Y_i - EY_i)(Y_j - EY_j)]\Big)_{i,j=1,2}$!

c) Calculate the correlation coefficient $\rho(Y_1, Y_2)$!

d) Find $E\Big((Y_2)^2 \,\Big|\, \sigma(X_1, X_2)\Big)$!

2.) (6 points) Let N_t, $t \geq 0$ is a homogeneous Poisson process with intensity $\lambda > 0$. Suppose $t_1 < t < t_2$. Calculate $p = P(N_{t_1} = 1\,,\ N_t = 3 \,|\, N_{t_2} = 4)$!

3.) (6 points) Let W_t, $t \geq 0$ be a standard Wiener process ($\sigma^2 = 1$). Define $X_t = 1 + W_t^2$, $t \geq 0$. Calculate $Cov(X_s, X_t)$ for $s < t$. **Hint:** Note $EW_u^4 = 3\,u^2$ for $u > 0$.

4.) (5 points) Let W_t, $t \geq 0$ be a standard Wiener process ($\sigma^2 = 1$). Define $Y_t = W_t\, e^{2\,W_t - 2\,t}$. Find dY_t !
Hint: Use Extension I of the Ito Lemma.

5.) (5 points) Put $Y_t = t + \ln(X_t) = f(t, X_t)$, $t \geq 0$, where

$$X_t = X_0 + \int_0^t (2\,s^2 - 1)\,X_s\,ds + \int_0^t 2\,s\,X_s\,dW_s \quad \text{with} \quad X_0 = 1\,,$$

where W_t, $t \geq 0$, is the standard Wiener process ($\sigma^2 = 1$). Find with Extension II of the Ito Lemma, that the process Y_t is a martingale with respect to the filtration, generated by $(W_t)_{t \geq 0}$.
Hint: Prove that the process Y_t is a Ito integral!

6.) (8 points) The following statements are TRUE or FALSE. So just answer TRUE, since ... or FALSE, since .... giving only a short explanation.

a) The random variables X_3 and Y_1 in **Problem 1** are uncorrelated.

b) Let $N(t)$, $t \geq 0$ is a homogenuous Poisson process with intensity $\lambda > 0$ and $t_1 < t < t_2$ (see **Problem 2**). Then $q = P(N(t_1) = 2\,,\ N(t) = 2 \,|\, N(t_2) = 5) = 0\,.$

c) The process X_t in **Problem 3** is a stationary process!

d) Let $N_t, t \geq 0$ be a Poisson process with intensity 2. The process $X_t = N_t - 4\,t$ is a martingale with respect to its natural filtration $\mathcal{F}_t = \mathcal{F}(N(u),\ 0 \leq u \leq t)$!

1.) (10 points) Suppose $X_1 \sim N(3,4)$, $X_2 \sim N(2,9)$, $X_3 \sim N(-1,1)$ and let X_1, X_2, X_3 be independent. Define $Y_1 = X_1 - 2X_2 + 1$ and $Y_2 = X_2 + X_3$.

a) Find the distribution of Y_1!

b) Calculate the covariance-matrix $\Sigma_{(Y_1,Y_2)} = \Big(E[(Y_i - EY_i)(Y_j - EY_j)]\Big)_{i,j=1,2}$!

c) Calculate the correlation coefficient $\rho(Y_1, Y_2)$.

d) Find $E(Y_1 Y_2 \big| \sigma(X_1, X_2))$.

2.) (6 points) Let $N(t)$, $t \geq 0$, is a homogeneous Poisson process with intensity $\lambda > 0$ and let X_1 denote the time of the first event. Calculate $P(X_1 \leq s\,,\, N(t) = 1 \big| N(u) = 2)$ for $0 \leq s < t < u$.

3.) (8 points) Let $W(t)$, $t \geq 0$ be a standard Wiener process. Show that $X_t = W_{t+3} - W_{t+1}$, $t \geq 0$, is a stationary process.

4. (6 points) Let Z be a random variable with $E|Z| < \infty$ and $\{\mathcal{F}_n\}_{n \geq 1}$ an increasing stream of information (filtration), i.e. $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Define $X_n = E\big(Z \big| \mathcal{F}_n\big)$, $n \geq 1$, which may be interpreted as successive forecasts. Prove that (X_n) is a martingale with respect to the filtration $\{\mathcal{F}_n\}$, $n \geq 1$, i.e. show $E\big(X_{n+1} \big| \mathcal{F}_n\big) = X_n$ for $n \geq 1$.

5.) (8 points) Let W_t, $t \geq 0$ be a standard Wiener process. Define

$$X_t = (W_t + t)\, e^{-(W_t + t/2)}, \quad t \geq 0\,.$$

Prove with Extension I of the Ito Lemma, that the process X_t is a martingale with respect to the filtration, generated by $(W_t)_{t \geq 0}$.

6.) (8 points) Let W_t, $t \geq 0$ be a standard Wiener process and let μ, σ, and r be positive constants. Consider the geometric Wiener process S_t given by the stochastic differential equation

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t\,, \quad S_0 = 1 \quad \text{and} \quad Y_t = f(t, S_t) = e^{-r\,t}\, S_t\,,$$

a) Find dY_t. (Hint: Use the Extension II of the Ito Lemma!)

b) Under which conditions on the parameters μ, σ, and r the process Y_t is a martingale with respect to the filtration, generated by $(W_t)_{t \geq 0}$.

7.) (8 points) The following statements are TRUE or FALSE. So just answer TRUE, since ... or FALSE, since .... giving only a short explanation.

a) Let Z_1 and Z_2 be identical distributed random variables with $EZ_k = 0$ and $\mathrm{Var}(Z_k) < \infty$, $k = 1, 2$. Then $Z_1 + Z_2$ and $Z_1 - Z_2$ are uncorrelated.

b) Let $N(t)$, $t \geq 0$ is a homogeneous Poisson process with intensity $\lambda \geq 0$. Suppose $t_1 < t_2 < t_3$. Then $P(N(t_2) = 3\,,\, N(t_3) = 3 \,\big|\, N(t_1) = 3) = 0\,.$

c) Let $N(t)$, $t \geq 0$ is a homogeneous Poisson process with intensity $\lambda \geq 0$. The process $Y_t = N(t + L) - N(L)$, $t \geq 0$, is a stationary process.

d) Let $W_t, t \geq 0$ be a Wiener process and let $A \sim N(2, 9)$. The process $X_t = At + W_t$ is a martingale!