$d$-dimensional symmetric bilinear DHO in $V(((1/r)d^2 + 3d + 2)/2, 2)$

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$d$-dimensional symmetric bilinear DHO in $V(((1/r)d^2 + 3d + 2)/2, 2)$

Let $d = lr \geq 4$.

We construct $d$-dimensional symmetric bilinear DHO $S_c$ in $V(((1/r)d^2 + 3d + 2)/2, 2)$ for $c \in GF(2^r)$ with $Tr(c) = 1$.

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Definition: (Huybrechts and Pasini, 1999)

*d-dimensional dual hyperoval* $S$ in $V(n, 2)$ is a set of $(d + 1)$-dim vector subspaces with:

1. $S$ consists of $2^{d+1}$ $(d + 1)$-subspaces,
2. any two distinct $(d + 1)$-subspaces of $S$ meet at one-dim subspace,
3. any three distinct $(d + 1)$-subspaces of $S$ intersect trivially,
4. $(d + 1)$-subspaces of $S$ span $V(n, 2)$.
If $d$-dimensional dual hyperoval in $V(n, 2)$ exists, it is considered that

$$2d + 1 \leq n \leq \frac{(d + 1)(d + 2)}{2}.$$

**Known $d$-dimensional DHO’s in** $V((d + 1)(d + 2)/2, 2)$.

- (1). Huybrechts’ DHO
- (2). Buratti and Del Fra’s DHO
- (3). Veronesean DHO [Thas, Van Maldeghem]
- (4). the deformation of Veronesean DHO [T]
Let $S_1$ be a $d$-dim. DHO in $V(n_1, 2)$ and $S_2$ a $d$-dim. DHO in $V(n_2, 2)$.

We say $S_1$ is a quotient of $S_2$, (or $S_2$ is a cover of $S_1$) if $\exists$ a surjective $GF(2)$-linear mapping (a covering map)

$$\pi : V(n_2, 2) \rightarrow V(n_1, 2) \text{ s.t. } \pi(S_2) = S_1.$$
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• konwn DHOs in $V(n, 2)$ for $2d + 2 < n < (d + 1)(d + 2)/2$ $\implies$ quotients of the four examples in $V(((d + 1)(d + 2)/2, 2)$.
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• Known DHOs in $V(n, 2)$ for $2d + 2 < n < (d + 1)(d + 2)/2 \implies$ quotients of the four examples in $V(((d + 1)(d + 2)/2, 2)$.

• Very recently, we construct examples of $d$-dim DHOs for $n = 3d + 3$ with $d \geq 3$, $n = 4d - 2$ with $d \geq 4$, and $n = 3d + 1$ with $3 \leq d \leq 13$ with some conditions, which are not quotients of the examples in $V(((d + 1)(d + 2)/2, 2)$. [FFA, 2013]
• Let $S_1$ be a $d$-dim. DHO in $V(n_1, 2)$ and $S_2$ a $d$-dim. DHO in $V(n_2, 2)$.
We say $S_1$ is a quotient of $S_2$, (or $S_2$ is a cover of $S_1$) if $\exists$ a surjective $GF(2)$-linear mapping (a covering map)

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• Known DHOs in $V(n, 2)$ for $2d + 2 < n < (d + 1)(d + 2)/2$ $\implies$ quotients of the four examples in $V((d + 1)(d + 2)/2, 2)$.

• In this talk, we construct many non-isomorphic symmetric bilinear DHOs for $n = ((1/r)d^2 + 3d + 2)/2$ ($d = lr$ with $lr \geq 4$), which are not quotients of the four DHOs in $V((d + 1)(d + 2)/2, 2)$. 
• Let $V$ be a $(d+1)$-dim. $GF(2)$-vector space, $W$ a $l$-dim. $GF(2)$-vector space. $S := \{X(e) \mid e \in V\}$ in $V \oplus W$ is a bilinear DHO if there is a $GF(2)$-bilinear mapping $B : V \oplus V \rightarrow W$ such that $X(e) := \{(x, B(x, e)) \mid x \in V\} \subset V \oplus W$ for any $e \in V$. 
• Let $V$ be a $(d+1)$-dim. $GF(2)$-vector space, $W$ a $l$-dim. $GF(2)$-vector space. $S := \{X(e) | e \in V\}$ in $V \oplus W$ is a bilinear DHO if there is a $GF(2)$-bilinear mapping $B : V \oplus V \to W$ such that $X(e) := \{(x, B(x, e)) | x \in V\} \subset V \oplus W$ for any $e \in V$.

• A bilinear DHO $S$ has a translation group $T := \{t_a | a \in V\}$, where

$$t_a(X(t)) = X(t + a)$$

defined by $t_a : V \oplus W \ni (x, y) \mapsto (x, y + B(x, a)) \in V \oplus W$. 
• Let \( V \) be a \((d+1)\)-dim. \( GF(2) \)-vector space, \( W \) a \( l \)-dim. \( GF(2) \)-vector space. \( S := \{ X(e) \mid e \in V \} \) in \( V \oplus W \) is a bilinear DHO if there is a \( GF(2) \)-bilinear mapping \( B : V \oplus V \to W \) such that \( X(e) := \{(x, B(x, e)) \mid x \in V \} \subset V \oplus W \) for any \( e \in V \).

• A bilinear DHO \( S \) has a translation group \( T := \{ t_a \mid a \in V \} \), where

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t_a(X(t)) = X(t + a)
\]

defined by \( t_a : V \oplus W \ni (x, y) \mapsto (x, y + B(x, a)) \in V \oplus W \).

• There are many bilinear DHOs, such as Huybrechts DHO, APN DHO, Buratti-Del Fra DHO, Yoshiara’s DHO. However, there exist also non-bilinear DHOs, such as Veronesean DHO, a deformation of Veronesean DHO.
• Let $V$ be a $(d+1)$-dim. $GF(2)$-vector space, $W$ a $l$-dim. $GF(2)$-vector space. $S := \{X(e) \mid e \in V\}$ in $V \oplus W$ is a bilinear DHO if there is a $GF(2)$-bilinear mapping $B : V \oplus V \rightarrow W$ such that $X(e) := \{(x, B(x, e)) \mid x \in V\} \subset V \oplus W$ for any $e \in V$.

• A bilinear DHO $S$ has a translation group $T := \{t_a \mid a \in V\}$, where
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defined by $t_a : V \oplus W \ni (x, y) \mapsto (x, y + B(x, a)) \in V \oplus W$.

• Our examples in $V(((1/r)d^2 + 3d + 2)/2, 2)$ for $d = lr \geq 4$, constructed in this talk, are symmetric and bilinear DHOs.
A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

Let $d := lr \geq 4$. $I = \{0, 1, \ldots, l\}$ and $I_0 = \{1, \ldots, l\}$

$V_1$: an $l$-dim. vector space over $GF(2^r)$ with a basis $\{e_1, \ldots, e_l\}$

$V_2$: an $(l + 1)$-dim. vector space over $GF(2^r)$ with a basis $\{e_0, e_1, \ldots, e_l\}$.

$V$: a $(rl + 1)$-dim. $GF(2)$-vector space generated by $V_1$ and $e_0$.

($V = V_1 \oplus \langle e_0 \rangle$ as $GF(2)$-vector space.)

$$V_1 \subset V \subset V_2.$$
A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

Let $c \in GF(2^r)$ s.t. $x^2 + (x/c) + 1 = 0$ has no solution in $GF(2^r)$.

Remark.

$\exists x \in GF(2^r)$ with $x^2 + (x/c) + 1 = 0$, put $y := cx$, we have $y^2 + y + c^2 = 0$, hence $Tr(c) = 0$.

Conversely, if $Tr(c) = 0$ with $c \neq 0$, $\exists y \in GF(2^r)$ with $y^2 + y + c^2 = 0$, hence $\exists x = y/c \in GF(2^r)$ with $x^2 + (x/c) + 1 = 0$.

Therefore, for $c \in GF(2^r)$, we have

$$x^2 + (x/c) + 1 \neq 0 \text{ for any } x \in GF(2^r) \iff Tr(c) = 1.$$
A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

Let $V_2 \otimes V_2 = \langle e_i \otimes e_j \mid i, j \in I \rangle$ over $GF(2^r)$. Inside $V_2 \otimes V_2$, let $W_c$ ($c$ is defined above) be a $GF(2^r)$-vector subspace generated by

$$(e_i \otimes e_j) + (e_j \otimes e_i) \text{ for all } i, j \in I \text{ with } i < j,$$

$e_0 \otimes e_0$ and $c(e_i \otimes e_i) + (e_0 \otimes e_i)$ for all $i \in I_0$. 
A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

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$$e_0 \otimes e_0 \text{ and } c(e_i \otimes e_i) + (e_0 \otimes e_i) \text{ for all } i \in I_0.$$

Let $x \otimes_c y$ the image of $x \otimes y + W_c$ in $(V_2 \otimes V_2)/W_c$. Then $x \otimes_c y = y \otimes_c x$ and $(x_1 + x_2) \otimes_c y = x_1 \otimes_c y + x_2 \otimes_c y$ for $x, x_1, x_2, y \in V_2$. In $(V_2 \otimes V_2)/W_c$,

$$x \otimes_c e_0 = e_0 \otimes_c x = (cx) \otimes_c x = x \otimes_c (cx) \quad (\forall x \in V_1) \text{ and } e_0 \otimes_c e_0 = 0.$$
A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

Let $V_2 \otimes V_2 = \langle e_i \otimes e_j \mid i, j \in I \rangle$ over $GF(2^r)$. Inside $V_2 \otimes V_2$, let $W_c$ ($c$ is defined above) be a $GF(2^r)$-vector subspace generated by

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\[e_0 \otimes e_0 \text{ and } c(e_i \otimes e_i) + (e_0 \otimes e_i) \text{ for all } i \in I_0.\]

Let $x \otimes_c y$ the image of $x \otimes y + W_c$ in $(V_2 \otimes V_2)/W_c$. Then $x \otimes_c y = y \otimes_c x$ and $(x_1 + x_2) \otimes_c y = x_1 \otimes_c y + x_2 \otimes_c y$ for $x, x_1, x_2, y \in V_2$. In $(V_2 \otimes V_2)/W_c$,

\[x \otimes_c e_0 = e_0 \otimes_c x = (cx) \otimes_c x = x \otimes_c (cx) \quad (\forall x \in V_1) \text{ and } e_0 \otimes_c e_0 = 0.\]

Hence $\{e_i \otimes_c e_j \mid 0 < i \leq j \leq l\}$ generate $(V_2 \otimes V_2)/W_c$. 
A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

Let $V_2 \otimes V_2 = \langle e_i \otimes e_j \mid i, j \in I \rangle$ over $GF(2^r)$. Inside $V_2 \otimes V_2$, let $W_c$ (c is defined above) be a $GF(2^r)$-vector subspace generated by

$$(e_i \otimes e_j) + (e_j \otimes e_i) \text{ for all } i, j \in I \text{ with } i < j,$$
$$e_0 \otimes e_0 \text{ and } c(e_i \otimes e_i) + (e_0 \otimes e_i) \text{ for all } i \in I_0.$$

Let $x \otimes_c y$ the image of $x \otimes y + W_c$ in $(V_2 \otimes V_2)/W_c$. Then $x \otimes_c y = y \otimes_c x$ and $(x_1 + x_2) \otimes_c y = x_1 \otimes_c y + x_2 \otimes_c y$ for $x, x_1, x_2, y \in V_2$. In $(V_2 \otimes V_2)/W_c$,

$$x \otimes_c e_0 = e_0 \otimes_c x = (cx) \otimes_c x = x \otimes_c (cx) \ (\forall x \in V_1) \text{ and } e_0 \otimes_c e_0 = 0.$$

Hence the image of natural injection $V_1 \otimes V_1 \rightarrow V_2 \otimes V_2$ generate $(V_2 \otimes V_2)/W_c$. 

A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

Let $V_2 \otimes V_2 = \langle e_i \otimes e_j \mid i, j \in I \rangle$ over $GF(2^r)$. Inside $V_2 \otimes V_2$, let $W_c$ ($c$ is defined above) be a $GF(2^r)$-vector subspace generated by

$$(e_i \otimes e_j) + (e_j \otimes e_i) \text{ for all } i, j \in I \text{ with } i < j,$$

$e_0 \otimes e_0$ and $c(e_i \otimes e_i) + (e_0 \otimes e_i)$ for all $i \in I_0$.

Let $x \otimes_c y$ the image of $x \otimes y + W_c$ in $(V_2 \otimes V_2)/W_c$.

**Proposition.** For non-zero $x, y \in V = V_1 \oplus \langle e_0 \rangle \subset V_2$, we have

$$x \otimes_c y = 0 \iff \begin{cases} (1) x = cy + e_0 \ (\not\in V_1) & \text{in case } y \in V_1, \\
(2) x = c^{-1}(y + e_0) \ (\in V_1) & \text{in case } y \not\in V_1 \text{ with } y \neq e_0, \\
(3) x = e_0 & \text{in case } y = e_0. \end{cases}$$
Lemma. Let $x, y \in V$ with $x, y \not\in V_1$. If $x \otimes_c y = 0$, then $x = y = e_0$.

Proof. Since $x, y \not\in V_1$, $x = x_0 + e_0$ and $y = y_0 + e_0$ with $x_0, y_0 \in V_1$. Assume $x \otimes_c y = 0$. Since $e_0 \otimes_c e_0 = 0$, $x \otimes_c y = x_0 \otimes_c y_0 + e_0 \otimes_c (x_0 + y_0) = x_0 \otimes_c y_0 + c(x_0 + y_0) \otimes_c (x_0 + y_0) = 0$. Let $x_0 = \sum_{i \in I_0} x_i e_i$, $y_0 = \sum_{i \in I_0} y_i e_i$ with $x_i, y_i \in GF(2^r)$. Then

$$x_0 \otimes_c y_0 + c(x_0 + y_0) \otimes_c (x_0 + y_0) = \sum_{i < j, i, j \in I_0} (x_i y_j + x_j y_i) (e_i \otimes_c e_j)$$

$$+ c \left( \sum_{i \in I_0} (x_i^2 + (x_i y_i/c) + y_i^2) (e_i \otimes_c e_i) \right) = 0.$$

Hence $x_i^2 + (x_i y_i/c) + y_i^2 = 0$ for any $i \in I_0$. Thus $(x_i, y_i) = (0, 0)$ for any $i \in I_0$, therefore $x_0 = y_0 = 0$, and $x = y = e_0$. QED
A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

We set $d := rl$. $V_1 = \langle e_1, \ldots, e_l \rangle \subset V = V_1 \oplus \langle e_0 \rangle \subset V_2 = \langle e_0, e_1, \ldots, e_l \rangle$.

Inside $V(((1/r)d^2 + 3d + 2)/2, 2) := V \oplus ((V_2 \otimes V_2)/W_c)$, for each $t \in V$, let us define a $(d + 1)$-dimensional vector subspace $X_c(t)$ by

$$X_c(t) := \{(x, x \otimes_c t) \mid x \in V\}.$$

**Theorem.** $S_c := \{X_c(t) \mid t \in V\}$ is a $d$-dimensional symmetric bilinear dual hyperoval in $V(((1/r)d^2 + 3d + 2)/2, 2)$.

**Proof.**

Since the image of $V_1 \otimes V_1$ generate $(V_2 \otimes V_2)/W_c$ and $V_1 \subset V$, $V(((1/r)d^2 + 3d + 2)/2, 2) = V \oplus ((V_2 \otimes V_2)/W_c)$ is generated by $\{(x, 0) \mid x \in V\} = X_c(0)$ and $\{(x, x \otimes_c t) \mid x \in V\} = X_c(t)$ for $t \in V \setminus \{0\}$. 
A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

For $s \neq t \in V$, let $X_c(s) \cap X_c(t) \ni (x, x \otimes_c s) = (x, x \otimes_c t)$ with $x \neq 0$. Then we have $x \otimes_c s = x \otimes_c t$, i.e., $x \otimes_c (s + t) = 0$.

We have (1) $x = c(s + t) + e_0$ ($\not\in V_1$) if $s + t \in V_1$, (2) $x = c^{-1}(s + t + e_0)$ ($\in V_1$) if $s + t \not\in V_1$ with $s + t \neq e_0$, and (3) $x = e_0$ if $s + t = e_0$.

For mutually distinct elements $s, t_1, t_2 \in V$, we have

$$X_c(s) \cap X_c(t_1) \neq X_c(s) \cap X_c(t_2).$$

Hence $X_c(s) \cap X_c(t_1) \cap X_c(t_2) = \{0\}$. There are $|V| = 2^{d+1}$ members in $S_c$.

Since $B(x, y) := x \otimes_c y$ is symmetric (i.e., $x \otimes_c y = y \otimes_c x$) and bilinear (i.e., $x \otimes_c (y_1 + y_2) = x \otimes_c y_1 + x \otimes_c y_2$), $S_c$ is a $d = lr$-dim. symmetric bilinear dual hyperoval in $V(((1/r)d^2 + 3d + 2)/2, 2)$. QED
A construction of $DHO$ $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

[If $r = 1$] \text{(i.e., $d = l$)}, then \[
\frac{(1/r)d^2 + 3d + 2}{2} = \frac{(d + 1)(d + 2)}{2}.
\]

Then $V = V_1 \oplus \langle e_0 \rangle$ is a $(d + 1)$-dim. vector space over $GF(2)$ with $V = V_2 = \langle e_0, e_1, \ldots, e_d \rangle$, $V_1 = \langle e_1, \ldots e_d \rangle$ and $c = 1 \in GF(2)$.

Inside $V \otimes V = \langle e_i \otimes e_j \rangle$, $W_c$ is a $GF(2)$-vector subspace generated by

\[
(e_i \otimes e_j) + (e_j \otimes e_i) \text{ for all } i, j \in I \text{ with } i < j,
\]

$e_0 \otimes e_0$ and $(e_i \otimes e_i) + (e_0 \otimes e_i)$ for all $i \in I_0$. 


Reconstruction of the Buratti-Del Fra DHO [Yoshiara, T, 2012].

Let $V$ be $(d + 1)$-dim. vector space over $GF(2)$.

$I := \{0 \leq i \leq d\}$, $I_0 := I \setminus \{0\}$. $V = \langle e_i \mid i \in I \rangle$, $V_1 = \langle e_i \mid i \in I_0 \rangle \subset V$.

Inside $V \otimes V = \langle e_i \otimes e_j \rangle$, let $W$ be a $GF(2)$-vector subspace generated by

\[ (e_i \otimes e_j) + (e_j \otimes e_i) \text{ for all } i, j \in I \text{ with } i < j, \]
\[ e_0 \otimes e_0 \text{ and } (e_i \otimes e_i) + (e_0 \otimes e_i) \text{ for all } i \in I_0. \]

Inside $(V \otimes V)/W$, $\overline{x \otimes y} = \overline{y \otimes x}$ and $\overline{(x_1 + x_2) \otimes y} = \overline{x_1 \otimes y} + \overline{x_2 \otimes y}$.

\[ \overline{x \otimes e_0} = \overline{e_0 \otimes x} = \overline{x \otimes x} \quad (\forall x \in V_1) \quad \text{and} \quad \overline{e_0 \otimes e_0} = 0. \]
Reconstruction of the Buratti-Del Fra DHO [Yoshiara, T, 2012].

Notice that, for $x, y \in V \setminus \{0\}$, we have

$$\overline{x \otimes y} = 0 \iff x = y + e_0 \text{ in case } y \neq e_0, \text{ and } x = e_0 \text{ in case } y = e_0.$$  

Let $A := V \oplus ((V \otimes V)/W) := \{(x, \bar{v}) \mid x \in V, v \in V \otimes V\}$.  

For each $t \in V$, we define a subset $X(t)$ of $A$ by

$$X(t) := \{(x, \overline{x \otimes t}) \mid x \in V\}.$$  

**Fact.** $S_B := \{X(t) \mid t \in V\}$ is the Buratti-Del Fra DHO in $V \oplus ((V \otimes V)/W) = V((d + 1)(d + 2)/2, 2)$.  

A construction of DHO $S_c$ for $c \in GF(2^r)$ with $Tr(c) = 1$

[ If $l = 1$ ] (i.e., $d = r$), then $((1/r)d^2 + 3d + 2)/2 = 2d + 1$.

$V = V_1 \oplus \langle e_0 \rangle$ as $GF(2)$-vector space, where $V_1 = \langle e_1 \rangle \cong GF(2^r)$.
$(V_2 \otimes V_2)/W_c \cong \langle e_1 \otimes_c e_1 \rangle$ as $GF(2^r)$-vector spaces.

For $x = x_0e_1 + \alpha e_0, y = y_0e_1 + \beta e_0 \in V$ with $x_0, y_0 \in GF(2^r), \alpha, \beta \in GF(2)$,

$$x \otimes_c y = c(\beta x_0^2 + x_0y_0/c + \alpha y_0^2)(e_1 \otimes_c e_1).$$

Inside $V(2d + 1, 2) := V \oplus ((V_2 \otimes V_2)/W_c)$, let us define

$$X_c(t) := \{(x, x \otimes_c t) \mid x \in V\}.$$ 

Then $S_c := \{X_c(t) \mid t \in V\}$ is a $d$-dim. symmetric bilinear DHO.
A dual hyperoval by U. Dempwolff (2013)

\[ d := r \geq 3. \ V = GF(2^r) \oplus \langle e_0 \rangle \text{ as } GF(2)\text{-vector space.} \]

For \( x = x_0 + \alpha e_0, y = y_0 + \beta e_0 \in V \) with \( x_0, y_0 \in GF(2^r) \) and \( \alpha, \beta \in GF(2) \),

\[ x \cdot y = \beta x_0^2 + cx_0y_0 + \alpha y_0^2 \in GF(2^r). \]

Inside \( V(2d + 1, 2) := (GF(2^r) \oplus \langle e_0 \rangle) \oplus GF(2^r) \), let us define

\[ X_c(t) := \{(x, x \cdot t) \mid x \in V\}. \]

Then \( S_c := \{X_c(t) \mid t \in V\} \) is a \( d \)-dim. symmetric bilinear DHO.
Fact. [Dempwolff and Edel]
Let $d \geq 3$. For $i = 1, 2$, let $S_i := \{X_i(e) \mid e \in V\}$, where

$$X_i(e) := \{(x, B_i(x, e)) \mid x \in V\} \subset V \oplus W$$

be $d$-dim. bilinear DHOs in $V \oplus W$ with $GF(2)$-bilinear mappings $B_i : V \oplus V \to W$. Then $S_1 \simeq S_2$ iff $\exists$ $GF(2)$-isom. $\lambda, \mu : V \to V$, $\rho : W \to W$, s.t. $\forall x, y \in V$

$$(B_1(x, y))^\rho = B_2(x^\lambda, y^\mu).$$
**Fact.** [Dempwolff and Edel]
Let \( d \geq 3 \). For \( i = 1, 2 \), let \( S_i := \{ X_i(e) \mid e \in V \} \), where

\[
X_i(e) := \{ (x, B_i(x, e)) \mid x \in V \} \subset V \oplus W
\]

be \( d \)-dim. bilinear DHOs in \( V \oplus W \) with \( GF(2) \)-bilinear mappings \( B_i : V \oplus V \to W \). Then \( S_1 \simeq S_2 \) iff \( \exists GF(2) \)-isom. \( \lambda, \mu : V \to V, \rho : W \to W \), s.t. \( \forall x, y \in V \)

\[
(B_1(x, y))^\rho = B_2(x^\lambda, y^\mu).
\]

**Theorem.** Let \( c_1, c_2 \in GF(2^r) \) s.t. \( Tr(c_1) = 1, Tr(c_2) = 1 \). Then

\[
S_{c_1} \simeq S_{c_2} \iff (x \otimes c_1 y)^\rho = x^\lambda \otimes c_2 y^\mu.
\]
Fact. [Dempwolff and Edel]
Let $d \geq 3$. For $i = 1, 2$, let $S_i := \{X_i(e) \mid e \in V\}$, where

$$X_i(e) := \{(x, B_i(x, e)) \mid x \in V\} \subset V \oplus W$$

be $d$-dim. bilinear DHOs in $V \oplus W$ with $GF(2)$-bilinear mappings $B_i : V \oplus V \to W$. Then $S_1 \simeq S_2$ iff $\exists$ $GF(2)$-isom. $\lambda, \mu : V \to V$, $\rho : W \to W$, s.t. $\forall x, y \in V$

$$(B_1(x, y))^\rho = B_2(x^\lambda, y^\mu).$$

Theorem. Let $c_1, c_2 \in GF(2^r)$ s.t. $Tr(c_1) = 1$, $Tr(c_2) = 1$. Then

$$S_{c_1} \simeq S_{c_2} \iff \exists \sigma \in Gal(GF(2^r)/GF(2)) \text{ s.t. } c_2 = c_1^\sigma.$$
$S_c$ is not a quotient of Veronesean DHO or deformation of Veronesean DHO

**Lemma.**

1. Let $S$ be a quotient of Veronesean dual hyperoval $S_V$ with the covering map $\pi : S_V \rightarrow S$. Then any $\phi \in Aut(S)$ fixes a member $\pi(X_V(\infty))$.

2. Let $S$ be a quotient of the deformation of Veronesean dual hyperoval $S_D$ with the covering map $\pi : S_D \rightarrow S$. Then any $\phi \in Aut(S)$ fixes two members $\pi(X_D(\infty))$ and $\pi(X_D(e_0))$. 
$S_c$ is not a quotient of Veronesean DHO or deformation of Veronesean DHO

**Lemma.**

1. Let $S$ be a quotient of Veronesean dual hyperoval $S_V$ with the covering map \( \pi : S_V \to S \). Then any \( \phi \in Aut(S) \) fixes a member \( \pi(X_V(\infty)) \).

2. Let $S$ be a quotient of the deformation of Veronesean dual hyperoval $S_D$ with the covering map \( \pi : S_D \to S \). Then any \( \phi \in Aut(S) \) fixes two members \( \pi(X_D(\infty)) \) and \( \pi(X_D(e_0)) \).

**Prop.** The bilinear dual hyperoval $S_c$ is not a quotient of the Veronesean dual hyperoval $S_V$, or the deformation of Veronesean dual hyperoval $S_D$. 
**Proposition.**
Let \( S \) be a quotient of Buratti-Del Fra DHO \( S_B \) or Huybrechts DHO \( S_H \).
Then \( \text{Aut}(S) < \text{Aut}(S_B) \) or \( \text{Aut}(S) < \text{Aut}(S_H) \).
Proposition.
Let $S$ be a quotient of Buratti-Del Fra DHO $S_B$ or Huybrechts DHO $S_H$. Then $\text{Aut}(S) < \text{Aut}(S_B)$ or $\text{Aut}(S) < \text{Aut}(S_H)$.

Proposition.
$S_B$ or $S_H$: Huybrechts DHO or Buratti-Del Fra DHO in $V \oplus W$
$= V((d + 1)(d + 2)/2, 2)$ with a bilinear mapping $B(x, t) : V \times V \to W$.

$S$: $d$-dimensional bilinear DHO in $V \oplus W_1$ with a bilinear mapping $B_1(s, t) : V \times V \to W_1$.

Let $S$ be a quotient of Buratti-Del Fra DHO $S_B$ or Huybrechts DHO $S_H$. Then, $\exists$ two linear isom. $\lambda, \mu : V \to V$, a linear surjection $\rho : W \to W_1$ s.t.

$$B_1(x^\lambda, t^\mu) = (B(x, t))^\rho.$$
$S_c$ is not a quotient of Buratti-Del Fra DHO $S_B$ if $c \neq 1$

Buratti-Del Fra DHO $S_B := \{X_B(t) \mid t \in V\}$, where $X_B(t) = \{(x, x \otimes t) \mid x \in V\} \subset V \oplus ((V \otimes V)/W)$ for any $t \in V$.

**Theorem.** *The dual hyperovals $S_c$ is not a quotient of $S_B$ if $c \neq 1$.\

We assume that $S_c$ is a quotient of $S_B$, and deduce a contradiction. Since $S_c$ is a quotient of $S_B$ by assumption, there exist two $GF(2)$-isomorphisms $\lambda, \mu : V \to V$ and a $GF(2)$-surjection $\rho : (V \otimes V)/W \to (V_2 \otimes V_2)/W_c$, such that

$$(\bar{x} \otimes \bar{y})^\rho = x^\lambda \otimes_c y^\mu.$$
$S_c$ is not a quotient of Buratti-Del Fra DHO $S_B$ if $c \neq 1$

Buratti-Del Fra DHO $S_B := \{X_B(t) \mid t \in V\}$, where $X_B(t) = \{(x, x \otimes t) \mid x \in V\} \subset V + ((V \otimes V)/W)$ for any $t \in V$.

**Theorem.** The dual hyperovals $S_c$ is not a quotient of $S_B$ if $c \neq 1$.

We assume that $S_c$ is a quotient of $S_B$, and deduce a contradiction. Since $S_c$ is a quotient of $S_B$ by assumption, there exist two $GF(2)$-isomorphisms $\lambda, \mu : V \to V$ and a $GF(2)$-surjection $\rho : (V \otimes V)/W \to (V_2 \otimes V_2)/W_c$, such that

$$(x \otimes y)^\rho = x^\lambda \otimes_c y^\mu.$$

If $c \neq 1$, (1) $\exists x \neq 0$ s.t. $x^\lambda = x^\mu = 0$, or (2) $\exists x \neq e_0$ s.t. $x^\lambda = x^\mu = e_0^\lambda = e_0^\mu = e_0$, which contradicts $\lambda$ and $\mu$ are $GF(2)$-isomorphisms.
$S_c$ is not a quotient of Huybrechts DHO

Huybrechts DHO is defined by $S_H := \{X_H(t) \mid t \in V\}$, where $X_H(t) := \{(x, x \wedge t) \mid x \in V\} \subset V \oplus (V \wedge V)$ for any $t \in V$.

Proposition. $S_c$ is not a quotient of Huybrechts dual hyperoval.

Proof.
Let $S_c$ be a quotient of Huybrechts DHO $S_H$. Then, there exist $\lambda, \mu : V \to V$ and a linear surjection $\rho : W = V \wedge V \to (V_2 \otimes V_2)/W_c$ s.t. $(x \wedge y)^\rho = x^\lambda \otimes_c y^\mu$.

Since $(x \wedge x)^\rho = x^\lambda \otimes_c x^\mu = 0$, we have $x^\lambda = c x^\mu + e_0$ for any non-zero $x \in V_1$ with $x^\mu \in V_1$, or we have $x^\lambda = c^{-1}(x^\mu + e_0)$ for any $x \in V_1$ with $x^\mu \not\in V_1$. Thank you very much!!
$S_c$ is not a quotient of Huybrechts DHO

Huybrechts DHO is defined by $S_H := \{ X_H(t) \mid t \in V \}$, where $X_H(t) := \{(x, x \wedge t) \mid x \in V\} \subset V \oplus (V \wedge V)$ for any $t \in V$.

**Proposition.** $S_c$ is not a quotient of Huybrechts dual hyperoval.

**Proof.** Let $S_c$ be a quotient of Huybrechts DHO $S_H$. Then, there exist $\lambda, \mu : V \to V$ and a linear surjection $\rho : W = V \wedge V \to (V_2 \otimes V_2)/W_c$ s.t. $(x \wedge y)^\rho = x^\lambda \otimes_c y^\mu$.

Since $(x \wedge x)^\rho = x^\lambda \otimes_c x^\mu = 0$, we have $x^\lambda = c x^\mu + e_0$ for any non-zero $x \in V_1$ with $x^\mu \in V_1$, or we have $x^\lambda = c^{-1}(x^\mu + e_0)$ for any $x \in V_1$ with $x^\mu \notin V_1$.

**A consequence.** If $c \neq 1$ and $l \geq 2, r \geq 2$, $S_c$ is new.
$S_c$ is not a quotient of Huybrechts DHO

Huybrechts DHO is defined by $S_H := \{X_H(t) \mid t \in V\}$, where $X_H(t) := \{(x, x \wedge t) \mid x \in V\} \subset V \oplus (V \wedge V)$ for any $t \in V$.

**Proposition.** $S_c$ is not a quotient of Huybrechts dual hyperoval.

**Proof.** Let $S_c$ be a quotient of Huybrechts DHO $S_H$. Then, there exist $\lambda, \mu : V \rightarrow V$ and a linear surjection $\rho : W = V \wedge V \rightarrow (V_2 \otimes V_2)/W$ s.t. $(x \wedge y)^{\rho} = x^\lambda \otimes_c y^\mu$.

Since $(x \wedge x)^{\rho} = x^\lambda \otimes_c x^\mu = 0$, we have $x^\lambda = cx^\mu + e_0$ for any non-zero $x \in V_1$ with $x^\mu \in V_1$, or we have $x^\lambda = c^{-1}(x^\mu + e_0)$ for any $x \in V_1$ with $x^\mu \notin V_1$.

**A consequence.** If $c \neq 1$ and $l \geq 2, r \geq 2$, $S_c$ is new.

Thank you very much!!