# Randomized Simplex Algorithms and Random Cubes
# — Extended Abstract —

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December 6, 1999

## Abstract

Despite its eminent practical impact, the general question for the theoretical behavior of the Simplex Algorithm is still open. However, in recent years some progress has been made by investigating randomized pivot rules. A main thread through the history of the analysis of the Simplex Algorithm has been the study of linear programs on combinatorial cubes.

In this paper, we propose two new randomized pivot rules. The behavior of both rules on the Klee-Minty cubes is analyzed completely. In the second part of the paper, we report on computer-based experiments (within the `polymake` framework) of different randomized pivot strategies on combinatorial cubes in low dimensions. In this context, we explore several concepts of random linear programs on cubes: random deformed products, random abstract objective functions, and random Matoušek functions.

AMS Subject Classification (1991): 90C05, 52B12, 68W20

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## 1 Introduction

The Simplex Algorithm in its geometric form is perhaps the most natural way one could proceed in order to maximize a linear functional $c \in (\mathbb{R}^d)^\star$ over the solutions $x \in \mathbb{R}^d$ of a system $Ax \leq b$ of linear inequalities. We may assume that the set of solutions is bounded, thus it is a *polytope*. Hence, an obvious algorithm would be to start at any vertex of the polytope and to proceed from vertex to vertex along (with respect to $c$) increasing edges until one reaches a maximum. For an introduction to the theory of polytopes, including the geometric aspects of the Simplex Algorithm, see Ziegler [16].

Many proposals have been made how to make the choices among the increasing edges at a vertex. Recipes for these choices are usually called *pivot rules*. Unfortunately, for none of them it could be proved that the resulting variant of the Simplex Algorithm performs a number of steps that is bounded polynomially in the (coding-)sizes of $A$, $b$, and $c$. Nevertheless, the question if *Linear Programming* is principally solvable in polynomial time was answered affirmatively by Khachiyan in 1979 [12]. However, it is still unknown if there is a *strongly polynomial* algorithm for Linear Programming. The ongoing search for polynomial pivot rules is probably the most common attempt to find such a strongly polynomial algorithm.

The first breakthrough into this direction was the analysis of a *randomized* pivot rule by Kalai [9] (and similarly by Matoušek, Sharir, and Welzl [15]). They proved that the expected number of steps can be bounded by a function which is subexponential in the dimension $d$ and the number $n$ of facets.

*Cubes* have come into focus in the context of the Simplex Algorithm because the examples of linear programs that fool some of the most famous pivot rules are deformed cubes (notably the *Klee-Minty cubes* "against" Bland's rule [2] and Dantzig's *largest coefficient rule*). Moreover, the existence of a "polynomial pivot rule" for combinatorial cubes (i.e., polytopes whose face lattices are isomorphic to the face lattices of cubes) is as unsettled as for general linear programs.

The goal of this paper is on the one hand to introduce two new pivot rules and to indicate their "theoretical potential" by analyzing their behaviors on the Klee-Minty cubes. The first new rule, called *random majority*, is an attempt to mimic the *greatest increase rule* (which for linear objective functions always proceeds to the neighbor with the largest objective function value) in a purely combinatorial way. The second rule, called *recursive random edge*, can be viewed as a variant of the well-known *random edge* rule. The latter rule, at every vertex, chooses one of the improving arcs uniformly at random. In spite of its simplicity, it has not yet been possible to analyze the behavior of that rule. Our variant modifies the rule in a way that allows to apply induction on the dimension in the analysis.

Furthermore, we present a computational study of various randomized pivot rules. Motivated by the importance of Linear Programming on cubes, our experiments are performed on different classes of (abstract) objective functions on cubes. The search for a fair choice of instances raises the question for the random generation of such objects. This will be discussed in Section 5.

The code for the experiments has been implemented within the `polymake`[1] framework by Gawrilow and Joswig [8]. The `polymake` system provides a platform for working with polytopes on a computer. It provides functions such as calculating the (directed) graph of a polytope, or computing the face lattice. Since it comes along with an easy to use C++-interface, it allows to write computer codes for experiments (like the ones we did) in a very convenient way.

## 2 Special Cubes

A more general concept than linear functions is that of *abstract objective functions* (*AOF*). An AOF on a polytope $P$ is an acyclic orientation of the graph of $P$ which has a unique sink in every subgraph induced by a (non-empty) face of $P$. This concept was considered by several authors, e.g. Kalai [10] and Gärtner [5].

In this paper, besides the problem of maximizing an arbitrary abstract objective function on a (combinatorial) cube, we will especially consider three types of abstract objective functions on cubes: the *Klee-Minty* examples, the *Matoušek* examples, and *deformed products*, where the latter two are generalizations of the first into two different directions.

A combinatorial way to describe the $d$-dimensional Klee-Minty cube is as follows (see [7]). Encode the vertices of the $d$-

[1] `polymake` is free software; it is available on the Internet at `http://www.math.tu-berlin.de/diskregeom/polymake/`.

dimensional Klee-Minty cube $\mathrm{KM}_d$ as 0/1-vectors of length $d$. For two 0/1-vectors $v, w \in \{0,1\}^d$ define $\mathrm{cp}(v,w)$ as the common prefix of $v$ and $w$. Clearly, the length $|\mathrm{cp}(v,w)|$ of the prefix $\mathrm{cp}(v,w)$ satisfies $0 \leq |\mathrm{cp}(v,w)| \leq d$, and $|\mathrm{cp}(v,w)| = d$ if and only if $v = w$. Moreover, let $(v)_2$ be the integer which is represented by $v$ in binary form, that is, $(v)_2 = \sum_{i=1}^{d} 2^{d-i} v_i$. Now define $v \prec w$ if $(v)_2 < (w)_2$ and the number of ones in $\mathrm{cp}(v,w)$ is even. Similarly, let $w \preceq v$ if $(v)_2 > (w)_2$ and the number of ones in $\mathrm{cp}(v,w)$ is odd. A geometric realization of the Klee-Minty cube can be obtained as a deformed product, see below.

The key property of the Klee-Minty cubes is that they have an ascending path from the bottom vertex 0 to the top vertex $e_1$ which passes through all the vertices. Another important feature is that each face of a Klee-Minty cube is a Klee-Minty cube itself.

*Deformed Products* are due to Amenta and Ziegler [1]. Let $P \subseteq \mathbb{R}^d$ be a polytope and $\varphi : P \longrightarrow [0,1]$ a linear function, mapping the polytope $P$ to the unit interval $[0,1]$. Let $V, W \subseteq \mathbb{R}^e$ be two normally equivalent polytopes. Then the *deformed product* $(P,\alpha) \bowtie (V,W)$ of $P$ and $(V,W)$ with respect to $\alpha$ is defined as

$$\left\{ \begin{pmatrix} v + \alpha(x)(w-v) \\ x \end{pmatrix} \;\middle|\; \begin{array}{l} v \in V, w \in W \\ x \in P \end{array} \right\} .$$

The essential property of the deformed product $(P,\alpha) \bowtie (V,W)$ is that it is combinatorially equivalent to the usual product $V \times P$ (or $W \times P$), [1, Theorem 3.4 (iii)]. In particular, if $P$ is a combinatorial $(d-1)$-cube and $V$ and $W$ are intervals, then $(P,\alpha) \bowtie (V,W)$ is a combinatorial $d$-cube.

Klee-Minty cubes now can be defined recursively as deformed products in the following way (see [1]). Let $\mathrm{KM}_0 := \{0\}$, $\mathrm{KM}_1 := [0,1]$, and for $d > 1$ (and an arbitrary $\varepsilon$ with $0 < \varepsilon < \frac{1}{2}$)

$$\mathrm{KM}_d := (\mathrm{KM}_{d-1}, x_1) \bowtie ([0,1], [\varepsilon, 1-\varepsilon]) \;,$$

where $x_1$ is the projection to the first coordinate.

While deformed products may in turn be viewed as generalizations of the geometric definition of Klee-Minty cubes, the abstract objective functions defined by Matoušek [14] generalize the purely combinatorial description of the Klee-Minty ordering of the cube vertices. To define such an abstract objective function take any upper triangular matrix $A \in \mathbb{F}_2^{d \times d}$ (where $\mathbb{F}_2$ is the field with two elements) with all diagonal elements equal to 1. Now identify the vertices of the (combinatorial) $d$-cube with $\mathbb{F}_2^d$, and let a vertex $v \in \mathbb{F}^d$ be *smaller* than another vertex $w \in \mathbb{F}_2^d$ (with respect to $A$) if $Av$ is *lexicographically smaller* than $Aw$, i.e., if $v_i < w_i$ holds for $i := \min\{i \mid v_i \neq w_i\}$. If we choose $A \in \mathbb{F}_2^{d \times d}$ to be the matrix with all entries on and above the main diagonal equal to 1, then we obtain the Klee-Minty ordering.

# 3 Randomized Pivot Rules

By a standard perturbation argument it can be shown that for investigations of the Simplex Algorithm one may restrict (at least theoretically) the attention to *simple* polytopes, i.e., $d$-dimensional polytopes with the property that every vertex is contained in precisely $d$ facets, or, equivalently, precisely $d$ edges meet at every vertex. In a simple polytope, every $k$-set of edges that meet in a common vertex span a $(k-1)$ dimensional face of the polytope.

Thus let $P$ be a $d$-dimensional simple polytope, endowed with an abstract objective function. In particular, the minimizing and the maximizing vertex are uniquely determined. We will denote them by $\mathrm{bot}_\varphi(P)$ and $\mathrm{top}_\varphi(P)$, respectively. This means that we only consider primal and dual non-degenerate linear programs.

Wishing to design a randomized pivot rule, the most natural choice probably is the following.

**Definition 3.1** *The* Random-Edge *(RE) pivot rule works as follows: For the current (non-optimal) vertex $v$ choose one of the outgoing arcs $(v, w)$ uniformly at random, and proceed to $w$.*

Unfortunately, it turns out that analyzing the behavior of RE is quite hard. This has been done successfully only for Matoušek's abstract objective functions and, in particular, for Klee-Minty cubes. A result of Kelly [11] implies that the expected number of pivot steps taken by RE on a $d$-cube with a Matoušek type abstract objective function (started at any vertex) is at bounded by $\binom{d+1}{2} = \mathrm{const} \cdot d^2$.

For Klee-Minty cubes, Gärtner, Henk, and Ziegler [7] have shown that this estimate is more or less sharp: If one chooses the starting vertex for RE uniformly at random, then the expected number of pivot steps will be at least

$$\frac{d^2}{4(H_{d+1} - 1)} ,$$

where $H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the $n-th$ *harmonic number*. Thus, there is a starting vertex of the $d$-dimensional Klee-Minty cube causing RE to perform at least $\mathrm{const} \cdot \frac{d^2}{\log d}$ pivot steps in expectation.

Random Facet is a randomized pivot rule that gives the Simplex Algorithm a recursive structure, thus allowing to analyze its behavior by induction on dimension.

**Definition 3.2** *The* Random-Facet *(RF) pivot rule works as follows: If the current (non-optimal) vertex $v$ has precisely one outgoing arc $(v, w)$, then proceed to $w$. Otherwise, choose one of the facets containing $v$ uniformly at random, say $F$, solve the problem restricted to $F$, and proceed from $\mathrm{top}_\varphi(F)$.*

Indeed, it turns out that employing recursion pays off (with respect to the possibility to analyze the behavior). Kalai ([9], see also [10]) has proved that for arbitrary (simple) $d$-dimensional polytopes with $n$ facets, the expected number of pivot steps (started at any vertex) is at most

$$2^{\mathrm{const}\sqrt{n \log d}} ,$$

which was the first subexponential upper bound on the running time of a variant of the Simplex Algorithm (another – similar – one is due to Matoušek, Sharir, and Welzl [15]).

Matoušek [14] has proved that for a randomly chosen one among his abstract objective functions on the $d$-cube RF is expected to perform at least

$$\mathrm{const} \cdot \frac{1}{d} 2^{\mathrm{const}\sqrt{d}}$$

pivot steps. Thus, at least for general abstract objective functions, RF does not lead to a polynomial Simplex Algorithm. However, Gärtner [6] proved that for those among Matoušek's abstract objective functions which actually can be obtained from a geometric realization of the $d$-cube along with some linear function the expected number of pivot steps taken by RF (starting at an arbitrary vertex) will be at most $\mathrm{const} \cdot d^2$.

A third rule which we consider in our computational study is a randomized version of Bland's rule. Bland [2] proposed the following (deterministic) rule: Before you start the Simplex Algorithm number the facets of the polytope in an arbitrary order. Within the algorithm, when you have to decide to which of the better neighbors

$w$ of a vertex $v$ you want to proceed, take the one with the property that the number of the (in case of a simple polytope uniquely determined) facet which does contain $v$ but not $w$ is as small as possible.

**Definition 3.3** *The* Random Bland *(RB) pivot rule works as follows: First choose a permutation of the facets of the polytope (uniformly) at random. Then run the Simplex Algorithm with Bland's (deterministic) rule.*

In this paper, we will consider two new pivot rules. The first one is a randomized pivot rule for the Simplex Method which, in principal, can be applied to arbitrary polytopes.

**Definition 3.4** *The* Random Majority *(RM) Pivot Rule works as follows. Among all the better neighbors choose one (uniformly at random) among those which maximize the in-degree.*

Thus, the RM rule works like RE, but on a restricted subgraph. Figure 5 in the Appendix shows this subgraph for a 4-cube endowed with some randomly chosen abstract objective function.

It should be noted that the proposed pivot rule is special in the sense that a single pivot step may be rather time-consuming. In fact, for each neighbor, the in-degree has to be computed; this means, that all the vertices at distance 2 must be visited. So the RM Pivot Rule makes sense, only if the set of vertices at distance 2 of a given vertex is reasonably small. For instance, for a simple $d$-polytope this is a set of cardinality at most $d(d-1)$.

The other new pivot rule that we consider is an attempt to modify RE in such a way that induction on dimension becomes possible.

**Definition 3.5** *The* Recursive Random Edge *(RRE) pivot rule works as follows: At the current (non-optimal) vertex $v$ determine the set $A$ of outgoing arcs. If $|A| = d$ (i.e., $v$ is the bottom vertex) choose an arc $(v, w) \in A$ uniformly at random and proceed to $w$. Otherwise, determine the $|A|$-dimensional face $F$ that is generated by $A$, solve the problem on $F$ (recursively, starting at $v$), and proceed then from the top-vertex of $F$.*

It is common to study the time complexity (with respect to some machine model) of an algorithm, or, in our context, of a pivot rule. Here, however, we are more concerned with the geometric ramifications of the subject.

**Definition 3.6** *Let $P$ be a polytope, $\varphi$ an abstract objective function on $P$, and $\mathcal{S}$ a randomized pivot strategy. The* maximal/average combinatorial complexity *of $\mathcal{S}$ on $P$ is defined as the maximum/average of the expected length of the path chosen by the Simplex Algorithm with start vertex $v$ using the strategy $\mathcal{S}$, where $v$ ranges over all vertices of $P$.*

The diameter of the graph of $P$ is a lower bound for the maximal combinatorial complexity for every choice of $\mathcal{S}$ and $\varphi$. This links the discussion to the still open Hirsch Conjecture, which says that the diameter is bounded by $n - d$. In Theorem 4.1 we show that on Klee-Minty cubes (even every deterministic variant of) the RM Pivot Rule has "optimal" combinatorial complexity for every start vertex. Theorem 4.2 states that on Klee-Minty cubes RRE has maximal combinatorial complexity at most $\text{const} \cdot d^2$.

## 4 Analysis of RM and RRE on the Klee-Minty Cubes

In the design of new pivot rules the main aspect is the question of how to analyze the rules. In fact, it seems conceivable that many of the known rules actually yield fast (maybe polynomial) variants of the Simplex Algorithm, but up to now nobody has managed to prove this. We analyze our new rules RM and RRE on the Klee-Minty cubes. It turns out that RM performs optimally.

**Theorem 4.1** *For every vertex $v$ of a Klee-Minty cube the expected path length for RM starting at $v$ equals the distance from $v$ to the top vertex in the graph.*

The maximal combinatorial complexity of RRE turns out to be bounded quadratically in the dimension. Notice that the average combinatorial complexity is the same as for RF, while the (sharp) estimate for the maximal combinatorial complexity differs from that for RF.

**Theorem 4.2** *For every vertex $v$ of the $d$-dimensional Klee-Minty cube the expected path length for RRE starting at $v$ is at most*

$$\begin{cases} \frac{d(d+2)}{4} & \text{if } d \text{ even} \\ \frac{(d+1)^2}{4} & \text{if } d \text{ odd} \end{cases} .$$

*Furthermore, the estimate is sharp for $v = 1 - e_1$.*

*The average combinatorial complexity of RRE on the $d$ dimensional Klee-Minty cube equals*

$$\frac{1}{4}\left(\binom{d+2}{2} - 1\right) .$$

The proofs of the theorems (which can be found in the Appendix) reveal techniques that may be also useful for an analysis of the behaviors of the rules for more general linear programs.

## 5 Random Cubes

Questions like "what is a random polytope?" or "what is a random linear program?" are of general interest, of course. For a more thorough discussion we refer to the full version of the paper. Here, we concentrate on the special issues of generating random deformed cubes, random general abstract objective functions, and random examples of Matoušek type.

We suggest to define a random deformed product of $d$ line segments as follows. Let $C_1 = [0,1]$ be the unit interval, and let $\alpha_1 : C_1 \to [0,1]$ be the identity map. For all $k$ with $1 < k \le d$, let $V_k = [0, v_k]$, $W_k = [x_k, y_k]$, where $v_k \in (0,1]$, $x_k, y_k \in [-k, k]$ are chosen uniformly at random (within the named intervals). Recursively define $C_k = (C_{k-1}, \alpha_{k-1}) \bowtie (V_k, W_k)$, where $\alpha_k$ is a direction (i.e. a point on the unit sphere, which is identified with a linear function) chosen uniformly at random and then affinely transformed such that it maps $C_k$ onto the interval $[0,1]$. As an objective function choose a direction uniformly at random. The definition of the intervals $V_k$ and $W_k$ is by no means canonical, but rather a deliberate choice.

The most general way to produce a random abstract objective function on a cube is based on the following property. Each abstract objective function on a simple polytope $P$ bijectively corresponds to a shelling order for the boundary of the dual $P^{\text{dual}}$, which is simplicial.

For convenience, suppose for the moment, that $\partial P^{\text{dual}}$ is extendibly shellable, that is, each partial shelling can be extended to a shelling of the whole boundary. Now one can produce a random shelling of $\partial P^{\text{dual}}$ as follows. Start by choosing a facet of $P^{\text{dual}}$ at random. This already constitutes a partial shelling. Then compute the set of all facets of $P^{\text{dual}}$ which extend this partial shelling to a partial shelling with one more facet. Choose one of these facets at random. We proceed until we are done. We cannot get stuck, because we assumed that $\partial P^{\text{dual}}$ is extendibly shellable.

Unfortunately, it is not known whether the boundary of the $d$-dimensional cross polytope, which is the dual of the $d$-cube, does satisfy our hypothesis. The good news is that it does not matter. Our model of a random abstract program on a cube just pretends that the cross polytope is extendibly shellable. If we are unlucky enough to end up in a non-extendible partial shelling then we just start over (after we saved the non-extendible partial shelling, of course, because very many people would find that interesting).

It should be stressed that we do not know anything about the distributions that are implicitly defined by our ways of generating random deformed products or random abstract objective functions.

Finally, there is another way to produce a random cube: Randomly choose an abstract cube program from Matoušek's class. This amounts to the random choice of a triangular $d \times d$-matrix with 0/1 entries. Note that this is a very restricted class.

# 6 Experimental Results

We investigated three classes of random linear/abstract programs on cubes: random deformed products, random abstract objective functions and random abstract objective functions of the Matoušek type. In order to complement our observations we also ran experiments on linear programs of spherical type. This is meant as a try to bridge the gap to Borgwardt's results, see below.

For each of the four classes of linear or abstract programs we created our samples as indicated in Section 5. For each linear or abstract program created and for each vertex of the respective polytope we computed the expected path lengths to the top vertex for the five randomized pivot strategies discussed in Section 3. These computations are exact: we employed the rational arithmetic implemented in the GNU Multiprecision Library (using `polymake`'s C++ wrapper classes). For each polytope and each strategy we computed the mean and the maximum values of the expected path lengths (taken over all the vertices), that is the average and the maximal combinatorial complexity, respectively.

The three types of random cubes have been studied in dimensions 3 to 10. For each dimension we created 100 samples of each type. Processing one 10-cube (with respect to all the pivot strategies) took about two hours on a SUN UltraSparc IIi (128MB/300MHz). Additionally, we listed the results for the Klee-Minty cubes.

For each type of cube, each dimension and each pivot strategy we computed the mean value of the average combinatorial complexities of all samples as well as the mean value of all maximal combinatorial complexities.

The data seem to indicate that the values of the average combinatorial complexity of RE grow more than linearly with the dimension. At least for the dimensions tested, the Klee-Minty examples are worse than all the other types considered. Note that Gärtner, Henk, and Ziegler [7] proved the growth indeed is quadratic for the Klee-Minty cubes; see also Section 3. The absolute values for RE are given in Tables 1 and Figure 7 in the Appendix.

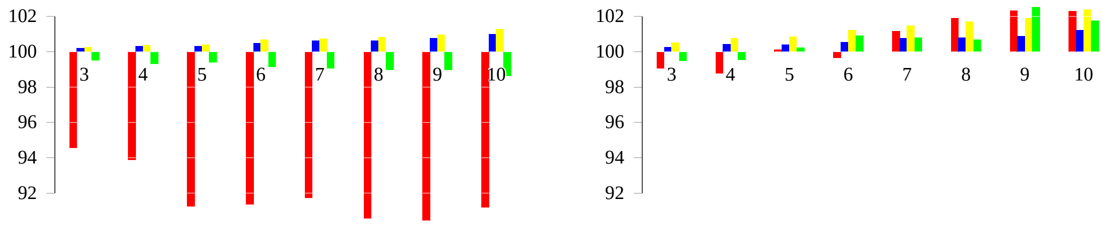


Figure 1: Deformed cubes. The numbers are percentage values; i.e. RE is set to 100. Therefore, it does not occur in the block chart. The blocks corresponding to the other pivot strategies are ordered in the following way: RM, RB, RF, RRE. This also holds for the other diagrams below.

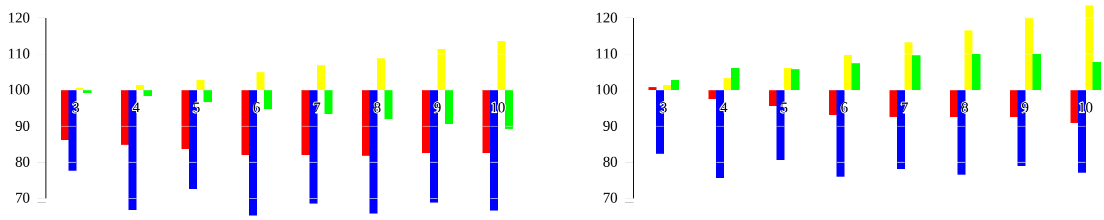


Figure 2: General AOF.

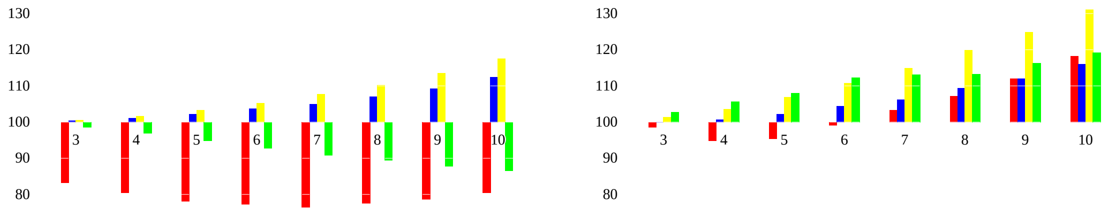


Figure 3: Matoušek type.

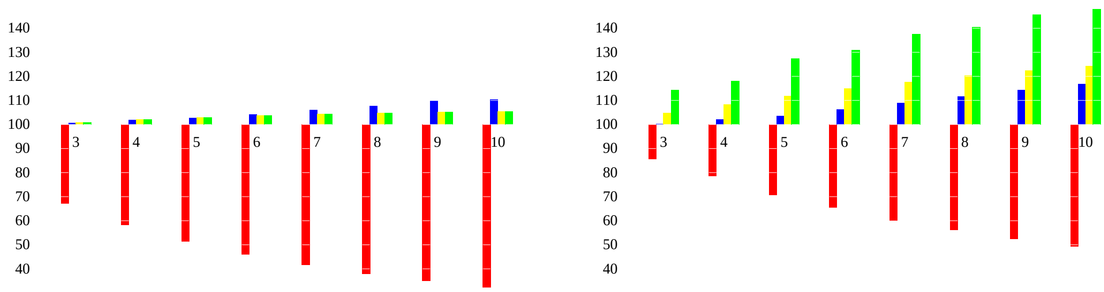


Figure 4: Klee-Minty cubes.

| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.58 | 2.14 | 2.83 | 3.39 | 3.96 | 4.69 | 5.17 | 5.87 |
| 2 | 1.87 | 2.62 | 3.35 | 4.21 | 5.07 | 5.84 | 6.58 | 7.25 |
| 3 | 1.89 | 2.77 | 3.85 | 5.17 | 6.58 | 7.99 | 10.02 | 11.93 |
| 4 | 2.22 | 3.43 | 4.85 | 6.51 | 8.38 | 10.50 | 12.84 | 15.43 |

| | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 1 | 3.02 | 4.04 | 5.12 | 6.17 | 7.19 | 8.33 | 9.15 | 10.48 |
| 2 | 3.24 | 4.47 | 5.67 | 7.02 | 8.38 | 9.63 | 10.90 | 12.09 |
| 3 | 3.17 | 4.46 | 5.97 | 7.73 | 9.58 | 11.45 | 13.93 | 16.31 |
| 4 | 3.50 | 5.08 | 7.06 | 9.16 | 11.62 | 14.23 | 17.17 | 20.27 |

Table 1: Mean values of the average (above) and maximal (below) combinatorial complexities of RE on different kinds of cubes. Cube Types: 1 = deformed cubes, 2 = general AOF, 3 = Matoušek type, 4 = Klee-Minty.

Instead of listing the values for the four other pivot strategies in further tables, for the sake of better comprehensibility, we compiled the information into block charts, see Figures 6–3. In these block charts we compare the mean values of the average combinatorial complexities and the mean values of the maximal combinatorial complexities, respectively, for each other pivot strategy with the corresponding values for RE.

Of course, one has to be careful with any interpretation of these results. Firstly, as already mentioned, we do not know much about the distributions of the deformed cubes and general AOFs generated. Secondly, the dimensions are rather low. Nevertheless, a few observations are immediate. RF seems to be the worst pivot strategy in almost all the cases. For the other pivot rules the patterns depend on the various types of cubes. With deformed cubes and Matoušek type AOFs RE is our winner for the maximal combinatorial complexities. For our samples of general AOFs RB is quite successful, both for the average and the maximal case. For the average combinatorial complexities RM is rather fast. The RRE Pivot Rule generally performs better than the original RE with respect to the average combinatorial complexity.

In order to explore whether the behavior of the pivot rules tested is special on cubes, we did the same computations also for other random polytopes than cubes. The random model which we used for these experiments relies on a special case of the distributions for which Borgwardt [3] proved that the Simplex Algorithm (with the deterministic *Shadow-Vertex* pivot rule) has an expected maximal combinatorial complexity of at most const $\cdot\, d \cdot n$. We generate these random linear programs (of *spherical type*) by choosing uniformly at random $n$ points on the $(d-1)$-dimensional unit-sphere in $\mathbb{R}^d$, whose convex hull is a simplicial polytope (almost surely). Thus the dual polytope is simple, and endowed with the projection onto the first coordinate as the linear objective function this gives us a random linear program.

For dimensions 3 to 6 we tried 100 random samples with 30, 40, 50 and 60 facets. For a low number of facets the behavior of all the pivot strategies is comparable to that on deformed cubes, both for the average and the maximal case. With an increasing number of facets the average case performance of RM decreases, while RRE improves. For the mean values of the maximal combinatorial complexities, however, RE seems to be unbeatable. The results of these experiments are collected in Table 2 and Figure 8.

## Acknowledgements

We are indebted to Ewgenij Gawrilow. Firstly, `polymake` would not exist without him. Secondly, he provided the perl code for producing the block charts in postscript. The necessary convex hull computations have been done with Fukuda's cdd [4].

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# Appendix

## Proof of Theorem 4.1

It is easy to compute the in-degree of a vertex of a Klee-Minty cube recursively. The bottom of the recursion is given by $\mathrm{indeg}(0) = 0$, where 0 here denotes the unique vertex of the 0-cube.

**Lemma 6.1** *The in-degree* $\mathrm{indeg}(v_1, \ldots, v_d)$ *of* $(v_1, \ldots, v_d)$ *equals* $d - \mathrm{indeg}(v_2, \ldots, v_d)$ *if* $v_1 = 1$ *and* $\mathrm{indeg}(v_2, \ldots, v_d)$ *otherwise.*

*Proof.* The set of vertices whose first coefficient is equal to 0, with the induced ordering, form a $(d-1)$-dimensional Klee-Minty cube. The set of vertices whose first coefficient is equal to 1, with the induced ordering, is a Klee-Minty cube with the ordering reversed. An easy induction completes the proof. □

The result of the theorem now immediately follows from induction and the lemma below.

**Lemma 6.2** *Let $v$ be any vertex of the $d$-dimensional Klee-Minty cube, and let $F$ be any facet containing $v$ and the top vertex $t$. Then applying the RM Pivot Rule to $v$ does not leave the facet $F$.*

*Proof.* Because the set of vertices whose first coefficient equals 1 is a Klee-Minty cube itself (with the reversed ordering), by induction, we can assume that $v = (0, v_2, \ldots, v_{k-1}, 0, v_{k+1}, \ldots, v_d)$. Here we suppose that the vertices of the facet $F$ have a 0 as the $k$-th coefficient and $k > 0$. Now it is sufficient to show the following: Whenever there is a vertex $w = (0, v_2, \ldots, v_{k-1}, 1, v_{k+1}, \ldots, v_d)$ with $v \prec w$, then there is some vertex $x \in F$ with $\mathrm{indeg}(x) > \mathrm{indeg}(w)$. As $v \prec w$ the number $n$ of 1-entries is even. We distinguish two cases.

Suppose that $n > 0$. Let $l$ be the position of the last 1 before position $k$, that is, $v = (0, v_2, \ldots, v_{l-1}, 1, 0, \ldots, 0, v_{k+1}, \ldots, v_d)$. Set

$$x = (0, v_2, \ldots, v_{l-1}, 0, 0, \ldots, 0, v_{k+1}, \ldots, v_d).$$

The common prefix of $v$ and $x$ has $n-1$ coordinates equal to 1. As $n-1$ is odd, we have $v \prec x$. Now $\mathrm{indeg}(x) - \mathrm{indeg}(w) = (d-l) - (d-k) = k - l > 0$.

Consider the case where $n = 0$, that is, $v = (0, \ldots, 0, v_{k+1}, \ldots, v_d)$. Set

$$x' = (1, 0, \ldots, 0, v_{k+1}, \ldots, v_d).$$

Clearly, $v \prec x'$ and $\mathrm{indeg}(x') - \mathrm{indeg}(w) = d - (d-k) > 0$. □

## Proof of Theorem 4.2

Let $P$ be any simple polytope and $\varphi$ an abstract objective function on $P$. For each vertex $v \in \mathrm{vert}(P)$ let $\mathrm{N}^{>}(v)$ be the set of neighbors $w$ of $v$ with $\varphi(w) > \varphi(v)$, let $\mathrm{F}^{>}$ be the $|\mathrm{N}^{>}(v)|$-dimensional face of $P$ spanned by $v$ and $\mathrm{N}^{>}(v)$, and let $\mathrm{top}(v) := \mathrm{top}(\mathrm{F}^{>}(v))$. By construction, we have $v = \mathrm{bot}(\mathrm{F}^{>}(v))$. In order to increase the readability, we omit the abstract objective function $\varphi$ from our notation.

The *bottom-top graph* of $P$ (with respect to $\varphi$) is the directed graph $\Psi(P)$ with vertices $\mathrm{vert}(P)$ and arcs

$$\{\, (v, \mathrm{top}(v)) \mid v \in \mathrm{vert}(P) \setminus \{\mathrm{top}(P)\}\} \ .$$

It is a tree in which all edges are oriented towards $\mathrm{top}(P)$, thus we consider the bottom-top graph as a tree rooted at $\mathrm{top}(P)$. We label the arcs $(v, w)$ by $\delta(v) := \dim(\mathrm{F}^{>}(v)) = |\mathrm{N}^{>}(v)|$. Fig. 6 shows the bottom top graph of a 4-cube with a randomly chosen abstract objective function.[2]

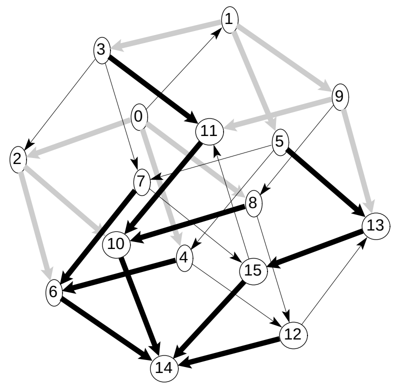


Figure 5: “RM graph” for a 4-dimensional cube with a randomly chosen abstract objective function. The thin arcs are those not considered by RM. The black arcs are the ones where RM does not have any choice, and the grey ones indicate that RM indeed has to make a (random) decision.

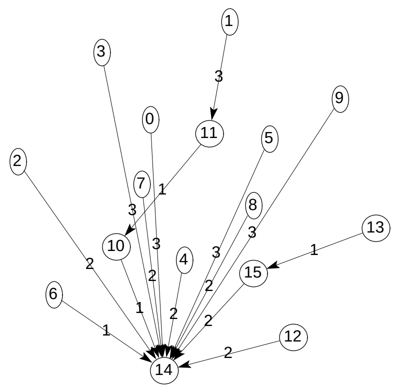


Figure 6: The bottom-top graph of the example from 5, where the labels on the arcs are the $\delta(\cdot)$-values.

Denote for every vertex $v \in \text{vert}(P)$ by $\tau(P, v)$ the expected path length for RRE on $P$ starting at $v$. If ($v = w_0, w_1, \ldots, w_r = \text{top}(P)$) is the path from $v$ to $\text{top}(P)$ in $\Psi(P)$, then we have

$$\tau(P, v) = \sum_{\alpha=0}^{r-1} \tau(\mathrm{F}^{>}(w_\alpha), w_\alpha) \qquad (1)$$

(with $w_\alpha = \text{bot}(\mathrm{F}^{>}(w_\alpha))$). Additionally, we have the following equation:

$$\tau(P, \text{bot}(P)) = \frac{\sum_{w \in \mathrm{N}^{>}(\text{bot}(P))} (1 + \tau(P, w))}{|\mathrm{N}^{>}(\text{bot}(P))|} \qquad (2)$$

From now on, let $P$ be the $d$-dimensional Klee-Minty Cube $\mathrm{KM}_d$ and $\varphi$ the coordinate function $x_1$. We identify the vertices of $\mathrm{KM}_d$ with $\mathbb{F}_2^d$. For any vertex $v \in \text{vert}(\mathrm{KM}_d)$ let

$$I(v) := \left\{ i \;\middle|\; v \oplus e_i \in \mathrm{N}^{>}(v) \right\}$$

(where we use $\oplus$ for the addition in $\mathbb{F}_2$) denote the set of coordinates along which one may proceed to larger neighbors of $v$. Then we have

$$I(v) = \{ i \mid v_1 + \cdots + v_i \equiv 0 \mod 2 \} \ .$$

From the definition of the Klee-Minty cube one can immediately read off $\text{top}(v)$ for every vertex $v \in \text{vert}(\mathrm{KM}_d)$:

$$\text{top}(v) = v \oplus e_i$$

with $i := \min I(v)$.

**Lemma 6.3** *For the $d$-dimensional Klee-Minty cube $\mathrm{KM}_d$ the equation*

$$\tau(\mathrm{KM}_d, \text{bot}(\mathrm{KM}_d)) = d$$

*holds.*

[2] Graphs drawn (via `polymake`) with Graphlet, Version 5.0, by F.J. Brandenburg et al., `http://www.fmi.uni-passau.de/Graphlet/`.

*Proof.* We proceed by induction on $d$, exploiting the fact that every face of a Klee-Minty cube is a Klee-Minty cube again. For $d = 1$ the claim is obvious, thus let $d > 1$. We have $I(\mathrm{bot}(\mathrm{KM}_d)) = I(0) = \{1, \ldots, d\}$ and for every $i \in \{1, \ldots, d\}$ the equations $I(e_i) = \{1, \ldots, i-1\}$, $\mathrm{top}(e_i) = e_1 \oplus e_i$, $I(e_1 \oplus e_i) = \{i, \ldots, d\}$, and $\mathrm{top}(e_1 \oplus e_i) = e_1 = \mathrm{top}(\mathrm{KM}_d)$. Thus, equations (2), (1), and the induction hypothesis imply

$$
\begin{aligned}
&\tau(\mathrm{KM}_d, \mathrm{bot}(\mathrm{KM}_d)) \\
&= \frac{1}{d}\sum_{i=1}^{d}(1 + \tau(\mathrm{KM}_d, e_i)) \\
&= \frac{1}{d}\left(1 + \sum_{i=2}^{d}(1 + (i-1) + (d-i+1))\right) \\
&= \frac{1}{d}(1 + (d-1)(d+1)) \\
&= d\ .
\end{aligned}
$$

□

Let $v \in \mathrm{vert}(\mathrm{KM}_d)$ be any vertex of $\mathrm{KM}_d$ with the $v$-$\mathrm{top}(v)$-path $(v = w_0, w_1, \ldots, w_r = \mathrm{top}(\mathrm{KM}_d))$ in $\Psi(\mathrm{KM}_d)$. Then, (1), (2), the fact that every face of a Klee-Minty cube is a Klee-Minty cube again, and Lemma 6.3 imply

$$\tau(\mathrm{KM}_d, v) = \sum_{\alpha=0}^{r-1} |I(w_\alpha)|\ . \qquad (3)$$

We have $w_{\alpha+1} = \mathrm{top}(w_\alpha)$ for each $\alpha = 0, \ldots, r-1$, thus $w_{\alpha+1} = w_\alpha \oplus e_i$ with $i := \min I(w_\alpha)$. Hence, $I(w_{\alpha+1}) = \{i+1, \ldots, d\} \setminus I(w_\alpha)$ holds, which yields $I(w_\alpha) \uplus I(w_{\alpha+1}) \subseteq \{i, \ldots, d\}$ (where '$\uplus$' means *disjoint* union) and $\min I(w_{\alpha+2}) \geq 2 + \min I(w_\alpha)$ for all $\alpha = 0, \ldots, r-2$. These facts lead to the following estimate:

$$
\begin{aligned}
\tau(\mathrm{KM}_d, v) &\leq |\{1, \ldots, d\}| + |\{3, \ldots, d\}| + \ldots \\
&= d + (d-2) + (d-4) + \ldots
\end{aligned}
$$

This sum can easily be evaluated, and one obtains the first result of the theorem.

In order to prove the second statement in the theorem, observe first that Equation (3) implies that the average combinatorial complexity $\tilde{\tau}(\mathrm{KM}_d)$ equals

$$\frac{\sum_{v \in V} \tau(\mathrm{KM}_d, v)}{2^d} = \frac{\sum_{w \in V'} \eta_w \cdot |I(w)|}{2^d}\ , \qquad (4)$$

where $V := \mathrm{vert}(\mathrm{KM}_d)$, $V' := V \setminus \{\mathrm{top}(\mathrm{KM}_d)\}$, and $\eta_w$ is the number of vertices in the subtree of $\Psi(\mathrm{KM}_d)$ (rooted at $\mathrm{top}(\mathrm{KM}_d)$) that is generated by $w$. In $\Psi(\mathrm{KM}_d)$ every vertex $w \neq \mathrm{top}(\mathrm{KM}_d)$ with $i := \min I(w)$ has in-degree $i - 1$, i.e., $w$ has $i - 1$ children in $\Psi(\mathrm{KM}_d)$ (this is due to the fact that for $w \neq \mathrm{top}(\mathrm{KM}_d)$ we have $I(w) \neq \varnothing$, and then a vertex $v$ has $\mathrm{top}(v) = w$ if and only if $v = w \oplus e_j$ for some $1 \leq j < i$). By the following induction, this observation yields $\eta_w = 2^{i-1}$ for all $w \in \mathrm{vert}(\mathrm{KM}_d)$ with $i = \min I(w)$. Indeed, for $i = 1$ this is clear since $w$ has no children in this case. For $i > 1$ the children $w_1, \ldots, w_{i-1}$ of $w$ satisfy

$$\{\min I(w_1), \ldots, \min I(w_{i-1}\} = \{1, \ldots, i-1\}\ .$$

Thus, the induction hypothesis implies

$$\eta_w = 1 + \sum_{j=1}^{i-1} 2^{j-1} = 1 + \sum_{j=0}^{i-2} 2^j = 2^{i-1}\ .$$

Hence, (4) equals

$$\frac{1}{2^d}\sum_{i=1}^{d} 2^{i-1}\omega_i$$

with $\omega_i := \sum |I(w)|$, where the sum ranges over all vertices $w \in \text{vert}(\text{KM}_d)$ with $\min I(w) = i$. For each $i$ we have $\min I(w) = i$ if and only if

$$w_1 = 1, w_2 = \cdots = w_{i-1} = 0, w_i = 1$$

hold. Since $I(\dots)$ defines a bijection between $\{w : w_1 = 1, w_2 = \cdots = w_{i-1} = 0, w_i = 1\}$ and the power set of $\{i+1, \dots, d\}$, we have

$$\begin{aligned} \omega_i &= \sum_{S \subseteq \{i+1,\dots,d\}} (1 + |S|) \\ &= 2^{d-i} + \sum_{s \subseteq \{i+1,\dots,d\}} |S| \\ &= 2^{d-i} + (d-i)2^{d-i-1} \\ &= (d-i+2)2^{d-i-1} \end{aligned}$$

(where the second last equation follows from counting incidences of elements of $\{i+1, \dots, d\}$ and subsets of $\{i+1, \dots, d\}$). Hence, we can calculate

$$\begin{aligned} \tilde{\tau}(\text{KM}_d) &= \frac{1}{2^d} \sum_{i=1}^{d} 2^{i-1}(d-i+2)2^{d-i-1} \\ &= \frac{1}{4} \sum_{i=1}^{d} (d-i+2) \\ &= \frac{1}{4} \left( \sum_{i=1}^{d+1} i - 1 \right) \\ &= \frac{1}{4} \left( \binom{d+2}{2} - 1 \right) \quad . \end{aligned}$$

Notice that this is precisely the same value as for RF, while the (sharp!) estimates of Theorem 4.2 differ from the corresponding results for RF.

## RE on Cubes

Mean values of the average combinatorial complexities of RE on different kinds of cubes.

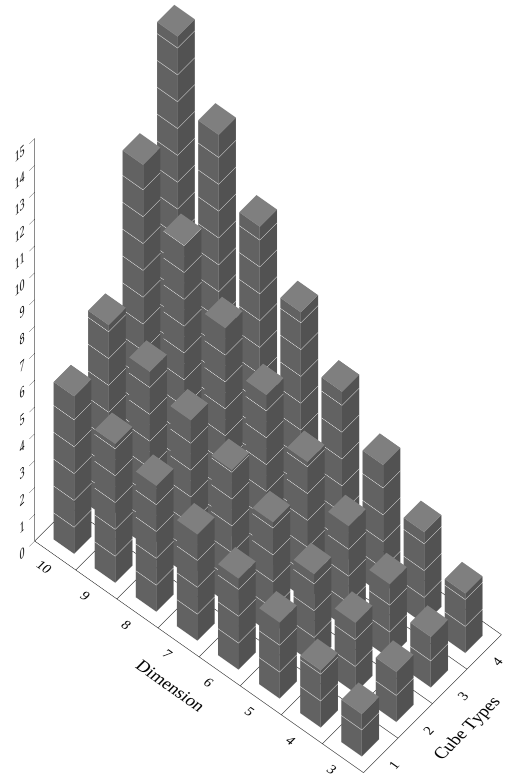


Figure 7: Cube Types: 1 = deformed cubes, 2 = general AOF, 3 = Matoušek type, 4 = Klee-Minty.

## Results on Polytopes of Spherical Type

| | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| 30 | 5.26 | 6.61 | 8.32 | 10.05 |
| 40 | 6.16 | 7.48 | 9.54 | 11.69 |
| 50 | 6.94 | 8.39 | 10.56 | 13.02 |
| 60 | 7.62 | 9.09 | 11.46 | 14.33 |

| | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| 30 | 5.26 | 6.61 | 8.32 | 10.05 |
| 40 | 6.16 | 7.48 | 9.54 | 11.69 |
| 50 | 6.93 | 8.39 | 10.56 | 13.02 |
| 60 | 7.62 | 9.09 | 11.46 | 14.33 |

Table 2: Mean values of the average and the maximal combinatorial complexities for RE on the spherical type.

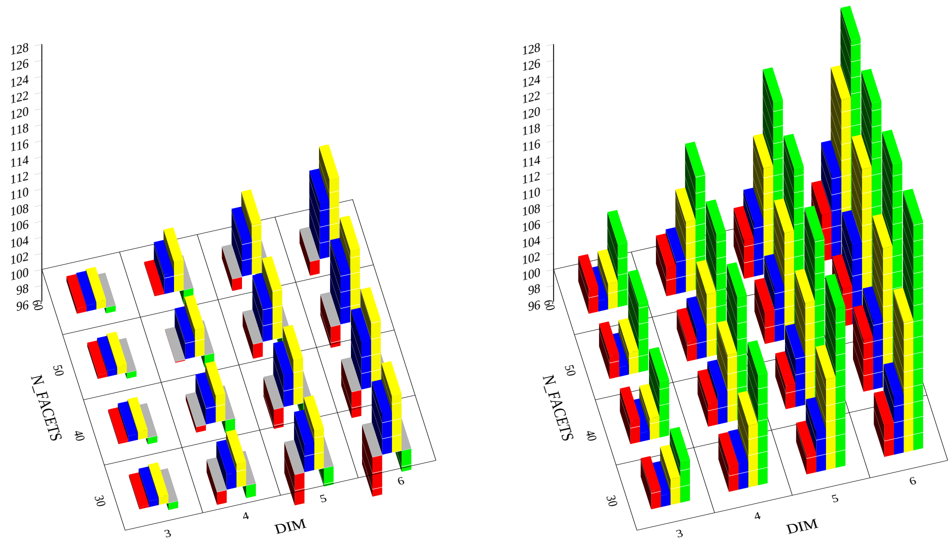


Figure 8: Percentage block charts for the spherical type. Average combinatorial complexity to the left, maximal combinatorial complexity to the right.