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ON THE SQAP-POLYTOPE*

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Abstract. The SQAP-polytope was associated to quadratic assignment problems with a certain symmetric objective function structure by Rijal (1995) and Padberg and Rijal (1996). We derive a technique for investigating the SQAP-polytope that is based on projecting the (low-dimensional) polytope into a lower dimensional vector-space, where the vertices have a "more convenient" coordinate structure. We exploit this technique in order to prove conjectures by Padberg and Rijal on the dimension of the SQAP-polytope as well as on its trivial facets.

Key words. quadratic assignment problem, symmetric model, polyhedral combinatorics

AMS subject classifications. 90C09, 90C10, 90C27

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1. Introduction.

For many classical $\mathcal{NP}$-hard combinatorial optimization problems like, e.g., the *traveling salesman problem (TSP)*, the *max cut problem*, or the *stable set problem*, the methods of polyhedral combinatorics have yielded a lot of structural insight that led to big improvements in practical problem solving via cutting-plane-based methods like branch&cut. However, the *quadratic assignment problem (QAP)*—where the task is to find a permutation π that minimizes $\sum_i \sum_k a_{ik} b_{\pi(i)\pi(k)} + \sum_i c_{i\pi(i)}$ for some matrices $A = (a_{ik})$, $B = (b_{jl})$, and $C = (c_{ij})$—was merely considered from a polyhedral point of view until the work of [24, 21] and [14] (which is a preliminary version of [16]). These papers defined the QAP-polytope via a well-known mixed integer programming (MIP) formulation of the QAP and proved some basic properties of that polytope, in particular its dimension (which was also proved in [5]).

There might be two reasons why the QAP-polytope had not been considered before. One is the fact that this polytope looks in some sense "nasty," which can be overcome by mapping it in a certain way into a different space (cf. [16]). The other reason is computational. The MIP-formulation on which the QAP-polytope is based has a lot of variables such that (at least) in former times it might have seemed impractical to solve the arising linear programs (LPs), for instance, within a branch&cut algorithm. However, the LP-solvers have improved a lot during the last few years, especially due to the success of interior point methods. Now, it seems promising to attack QAP-instances of size about 20 or 25 (and maybe even larger) by cutting-plane-based algorithms that use structural insight into the QAP-polytope. When considering these orders of magnitudes, one has to note that existing branch&bound algorithms (mostly using the *Gilmore–Lawler bound*) need a large amount of (parallel) computer power to solve instances of size about 20, since they produce branch&bound trees with a lot of nodes (cf. [8]). In the meantime during the submission of this paper and the preparation of the final version, powerful branch&bound codes have been developed that rely on more elaborate lower bounding procedures [12, 6]. Nevertheless, these algorithms also produce large branch&bound trees. Due to this fact, it

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sounds attractive to try to reduce this "tendency to implicit enumeration" by exploiting structural information about the problem that results from the polyhedral investigations.

Actually, the kind of QAP we defined above is a so-called *Koopmans–Beckmann problem (KB-QAP)*. It was introduced in [19] in order to model the situation of a set of n facilities that have certain amounts of "flow" between them and a set of n locations having certain distances, and the requirement is to assign the facilities to the locations in such a way that the sum of the products of flows and the respective distances is minimized. The c_{ij} model fixed costs that arise when placing facility i to location j independently from the assignment of the other facilities. One calls matrix A the *flow matrix*, matrix B the *distance matrix*, and matrix C the matrix of the *linear costs*. Clearly, this problem is $\mathcal{NP}$-hard, since it has many $\mathcal{NP}$-hard optimization problems as special cases, e.g., the TSP.

We call *symmetric* such instances with the property that assigning object i to location j and object k to location l always causes the same costs as assigning i to l and k to j. For example, all instances having a symmetric distance or flow matrix are symmetric in that sense. It turns out (first observed by [24, 21]) that for such symmetric instances one can drop nearly 50% of the variables in the MIP-formulation underlying the polyhedral approach. (Doing this, the quality of the LP-relaxation decreases slightly, as we will show in section 6.) This yields a different polytope, the symmetric QAP-polytope (SQAP-polytope). In [24] and [21] a set of valid equations for that polytope is derived and the dimension of the SQAP-polytope is conjectured.

In this paper, we present some basic properties of the SQAP-polytope including a proof of that conjecture. The main tool we use is a transformation that is similar to the one that allowed us to derive basic results about the QAP-polytope in a (relatively) simple way [16]. In section 2 we explain a formulation of the QAP as a minimization problem in a certain graph. Using that terminology, we give the MIP-formulations for QAP and SQAP that underlie the polyhedral approaches. In section 3 we give definitions of both the QAP- and SQAP-polytopes and describe connections between them. Then we map these polytopes isomorphically to other spaces, where they "look much nicer." (When saying a certain polytope P is *isomorphic* to a polytope P', we always mean that there is an affine transformation from $\operatorname{aff}(P)$ to $\operatorname{aff}(P')$ mapping P to P'. In particular, this implies that the two polytopes are combinatorially isomorphic, i.e., they have isomorphic face lattices.) In section 4 the dimension of the SQAP-polytope as well as the fact that the nonnegativity constraints on the variables define facets of it are established. In section 5 we present a first class of nontrivial facets of the SQAP-polytope. Some computational results concerning a lower bound obtained by exploiting these first results about the SQAP-polytope are reported in section 6. Finally we give some conclusions in section 7.

2. Problem definition. We will define the QAP as the problem of finding among certain cliques in a special graph one of minimal node and edge weight. The SQAP will be defined as a similar problem in a closely related hypergraph. We use the symbol $\binom{M}{k}$ for the set of all subsets of cardinality k of a set M.

Let the graph $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ have nodes

$$\mathcal{V}_n := \big\{(i,j) \;\big|\; i,j \in \{1,\dots,n\}\big\}$$

and edges

$$\mathcal{E}_n := \big\{\{(i,j),(k,l)\} \in \tbinom{\mathcal{V}_n}{2} \;\big|\; i \neq k, j \neq l\big\}.$$

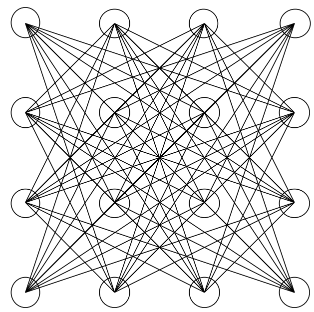

FIG. 2.1. *The graph $\mathcal{G}_n$ has all possible edges except the "horizontal" and the "vertical" ones.*

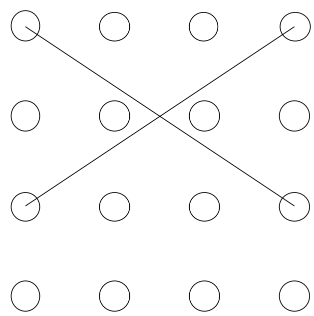

FIG. 2.2. *A pair of edges that can be identified in the symmetric case.*

We define $[i,j,k,l] := \{(i,j),(k,l)\}$ for all edges $\{(i,j),(k,l)\} \in \mathcal{E}_n$. This implies $[i,j,k,l] = [k,l,i,j]$. We usually draw $\mathcal{G}_n$ as shown in Figure 2.1.

The graph $\mathcal{G}_n$ has clique-number $\omega(\mathcal{G}_n) = n$, and the n-cliques of $\mathcal{G}_n$ correspond to the $(n \times n)$-permutation matrices. We denote the set of (node sets of) k-cliques of $\mathcal{G}_n$ by

$$\mathcal{CLQ}_k^n := \{C \subseteq \mathcal{V}_n \mid C \ k\text{-clique of } \mathcal{G}_n\}.$$

For any $S \subseteq \mathcal{V}_n$, we denote by $\mathcal{E}_n(S) := \{\{v,w\} \in \mathcal{E}_n \mid v,w \in S\}$ the set of edges having both endpoints in S. As usual, for a subset $N \subseteq M$ of a finite set M and a vector $a \in \mathbb{R}^M$, we define $a(N) := \sum_{e \in N} a_e$.

The QAP is to solve

$$(\mathrm{QAP}_{g,h}) \qquad \begin{array}{rl} \min & g(C) + h(\mathcal{E}_n(C)) \\ \text{subject to} & C \in \mathcal{CLQ}_n^n \end{array}$$

for given node weights $g \in \mathbb{R}^{\mathcal{V}_n}$ and edge weights $h \in \mathbb{R}^{\mathcal{E}_n}$. (If we have a KB-QAP defined by the matrices $A = (a_{ik})$, $B = (b_{jl})$, and $C = (c_{ij})$, we choose $g_{(i,j)} = c_{ij} + a_{ii}b_{jj}$ and $h_{[i,j,k,l]} = a_{ik}b_{jl} + a_{ki}b_{lj}$.)

The nodes and edges of $\mathcal{G}_n$ will correspond to variables in the polyhedral approach. If the instance (g,h) is *symmetric* in the sense that $h_{[i,j,k,l]} = h_{[i,l,k,j]}$ for all pairs of edges $\{[i,j,k,l],[i,l,k,j]\}$ (cf. Figure 2.2), then we can identify these two edges in our formulation, and hence reduce the number of variables by nearly 50%.

This observation (first made by [24, 21]) gives the motivation to study also a specific formulation for the special case of symmetric instances of the QAP, the *SQAP*.

In order to derive an appropriate formulation for SQAP, we model the described identification of edges by passing from the graph $\mathcal{G}_n$ having nodes $\mathcal{V}_n$ and edges $\mathcal{E}_n$ to the hypergraph $\mathcal{H}_n$ having the same nodes $\mathcal{V}_n$, but hyperedges

$$\mathcal{F}_n := \left\{\{(i,j),(k,l),(i,l),(k,j)\} \in \tbinom{\mathcal{V}_n}{4} \middle| i \neq k, j \neq l\right\}.$$

There will be no hypergraph theory involved; we simply use the notions of "hypergraph" and "hyperedges." For $i \neq k$ and $j \neq l$, we write $\langle i,j,k,l\rangle := \{(i,j),(k,l),(i,l),(k,j)\}$. This implies $\langle i,j,k,l\rangle = \langle k,l,i,j\rangle = \langle i,l,k,j\rangle = \langle k,j,i,l\rangle$ for all $i \neq k$ and $j \neq l$. For an edge $[i,j,k,l] \in \mathcal{E}_n$ we call the edge $\tau([i,j,k,l]) := [i,l,k,j]$ the *mate* of $[i,j,k,l]$. Then we can assign to every edge $e \in \mathcal{E}_n$ a hyperedge $\mathrm{HYP}(e) := e \cup \tau(e) \in \mathcal{F}_n$. For a subset $R \subseteq \mathcal{E}_n$, we denote $\mathrm{HYP}(R) := \{\mathrm{HYP}(e) \mid e \in R\}$. For a subset $S \subseteq \mathcal{V}_n$, we define the set $\mathcal{F}_n(S) := \mathrm{HYP}(\mathcal{E}_n(S))$. We refer to a subset $C \subset \mathcal{V}_n$ as a *clique of* $\mathcal{H}_n$ if and only if C is a clique of the graph $\mathcal{G}_n$.

Because we need to express relationships between the asymmetric and the symmetric versions of the problem, we introduce the map

$$\mathrm{sym}_n \;:\; \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$$

by defining $\mathrm{sym}_n(x,y) = (x,z)$ via $z_{e\cup\tau(e)} := y_e + y_{\tau(e)}$ for all $e \in \mathcal{E}_n$.

If $(g,h) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ and h is symmetric, then $(\mathrm{QAP}_{g,h})$ is equivalent to solving SQAP

$$(\mathrm{SQAP}_{g,\widehat{h}}) \qquad \begin{array}{rl} \min & g(C) + \widehat{h}(\mathcal{F}_n(C)) \\ \text{subject to} & C \in \mathcal{CLQ}_n^n \end{array}$$

with $\widehat{h}_{\mathrm{HYP}(e)} := h_e$ for all $e \in \mathcal{E}_n$.

In the rest of this section, we will develop MIP-formulations for QAP and SQAP. These formulations are the starting points for the polyhedral approach. The MIP-formulation for QAP was introduced by [13] and [1] (using a general linearization technique due to [2]). It is similar to a formulation by [9], which, however, was demonstrated by [13] and [1] to give a weaker LP-relaxation. The one for SQAP is due to [24] and [21]. Nevertheless, we will give short proofs of the respective theorems in our notational setting.

We need the notion of a *characteristic vector* $\chi^N \in \{0,1\}^M$ for a subset $N \subseteq M$ of a (finite) set M, defined by setting $\chi^N_p := 1$ for $p \in M$ if and only if $p \in N$. We will denote characteristic vectors of subsets of

$$\begin{array}{lll} \mathcal{V}_n & \text{by} & x^{(\cdots)}, \\ \mathcal{E}_n & \text{by} & y^{(\cdots)}, \text{ and} \\ \mathcal{F}_n & \text{by} & z^{(\cdots)}. \end{array}$$

Define $\mathrm{VERT}_n := \{(x^C, y^{\mathcal{E}_n(C)} \mid C \in \mathcal{CLQ}_n^n\}$ and $\mathrm{SVERT}_n := \{(x^C, z^{\mathcal{F}_n(C)} \mid C \in \mathcal{CLQ}_n^n\}$, i.e., VERT_n and SVERT_n are the characteristic vectors of feasible solutions to QAP and SQAP, respectively.

We denote by $\mathrm{row}_i^{(n)} := \{(i,j) \in \mathcal{V}_n \mid j = 1,\ldots,n\}$ the ith row and by $\mathrm{col}_j^{(n)} := \{(i,j) \in \mathcal{V}_n \mid i = 1,\ldots,n\}$ the jth column of the nodes $\mathcal{V}_n$. The next two theorems provide the desired MIP-formulations for QAP and SQAP, respectively. As usual, for any two disjoint subsets $S,T \subseteq \mathcal{V}_n$, $(S : T)$ is the set of all edges in $\mathcal{E}_n$ having one endpoint in S and the other one in T. For a singleton $\{v\}$, in this as well as in some other contexts, we often omit the brackets and simply write v.

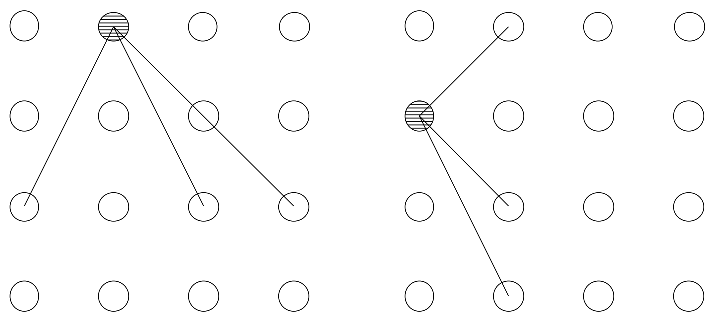

FIG. 2.3. *Equations* (2.3) *and* (2.4).

Figures 2.3 and 2.4 illustrate the used equations. We draw a hyperedge from $\mathcal{F}_n$ simply by drawing both mates from $\mathcal{E}_n$ belonging to that hyperedge. In all our figures, dashed nodes or (hyper)edges indicate coefficients -1, solid ones stand for $+1$.

THEOREM 2.1. *A vector* $(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ *is a member of* VERT_n *if and only if it satisfies the following conditions:*

$$x(\mathrm{row}_i^{(n)}) = 1 \quad (i = 1, \ldots, n), \tag{2.1}$$

$$x(\mathrm{col}_j^{(n)}) = 1 \quad (j = 1, \ldots, n), \tag{2.2}$$

$$-x_{(i,j)} + y((i,j) : \mathrm{row}_k^{(n)}) = 0 \quad (i, j, k = 1, \ldots, n,\ i \neq k), \tag{2.3}$$

$$-x_{(i,j)} + y((i,j) : \mathrm{col}_l^{(n)}) = 0 \quad (i, j, l = 1, \ldots, n,\ j \neq l), \tag{2.4}$$

$$y_e \geq 0 \quad (e \in \mathcal{E}_n), \tag{2.5}$$

$$x_v \in \{0, 1\} \quad (v \in \mathcal{V}_n). \tag{2.6}$$

We make one more notational convention in order to increase the readability of the following equations. For any pair $v, w \in \mathcal{V}_n$ of nodes belonging to the same row or column of $\mathcal{V}_n$, we denote by $\Delta_v^w := \{f \in \mathcal{F}_n \mid v, w \in f\}$ the set of all hyperedges in $\mathcal{F}_n$ containing both v and w (cf. Figure 2.4).

THEOREM 2.2. *A vector* $(x, z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ *is a member of* SVERT_n *if and only if it satisfies the following conditions:*

$$x(\mathrm{row}_i^{(n)}) = 1 \quad (i = 1, \ldots, n), \tag{2.7}$$

$$x(\mathrm{col}_j^{(n)}) = 1 \quad (j = 1, \ldots, n), \tag{2.8}$$

$$-x_{(i,j)} - x_{(k,j)} + z(\Delta_{(i,j)}^{(k,j)}), = 0 \quad (i, j, k = 1, \ldots, n, i < k), \tag{2.9}$$

$$-x_{(i,j)} - x_{(i,l)} + z(\Delta_{(i,j)}^{(i,l)}), = 0 \quad (i, j, l = 1, \ldots, n, j < l), \tag{2.10}$$

$$z_e \geq 0 \quad (e \in \mathcal{F}_n), \tag{2.11}$$

$$x_v \in \{0, 1\} \quad (v \in \mathcal{V}_n). \tag{2.12}$$

Proof of Theorem 2.1. The "only if" part is clear. To see the other direction, let $(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ satisfy conditions (2.1)–(2.6). Obviously, x is the characteristic vector of an n-clique of $\mathcal{G}_n$, and one deduces (e.g., using two equations from (2.3) and the nonnegativity of y) that $y_{[i,j,k,l]} > 0$ implies $x_{(i,j)} = x_{(k,l)} = 1$. These two facts imply that it is impossible for two components of y belonging to mates to be both nonzero. Observing that $\mathrm{sym}_n(x, y)$ satisfies the conditions of Theorem 2.2, one obtains Theorem 2.1 from Theorem 2.2. $\square$

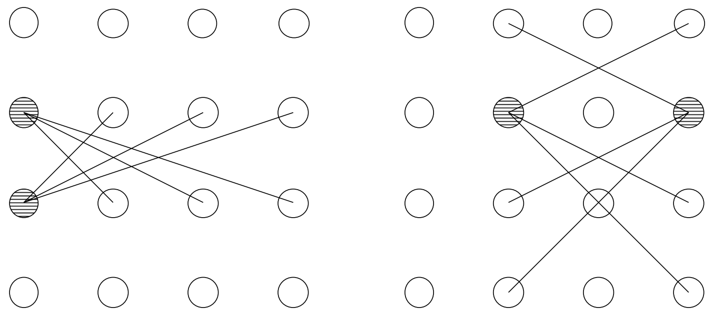

FIG. 2.4. *Equations* (2.9) *and* (2.10).

Proof of Theorem 2.2. Again, the "only if" part is obvious. Let $(x, z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ satisfy conditions (2.7)–(2.12); hence x is the characteristic vector of an n-clique $C \in \mathcal{CLQ}_n^n$. Considering four appropriate equations from (2.9) and (2.10) (and noting the nonnegativity of z), one gets that $z_{\langle i,j,k,l\rangle} > 0$ implies $x_{(i,j)} = x_{(k,l)} = 1$ or $x_{(i,l)} = x_{(k,j)} = 1$. But then, in each of (2.9) and (2.10), there is at most one hyperedge involved corresponding to a nonzero component of z. This leads to the fact that $z_{\langle i,j,k,l\rangle} > 0$ implies $z_{\langle i,j,k,l\rangle} = 1$, and that $x_{(i,j)} = x_{(k,l)} = 1$ implies $z_{\langle i,j,k,l\rangle} = 1$. Hence, z must be the characteristic vector of $\mathcal{F}_n(C)$. ∎

3. The SQAP-polytope and some relatives. Theorems 2.1 and 2.2 give us the starting points for deriving and exploiting further structural information on the problems QAP and SQAP. As with many other combinatorial optimization problems, the hope is to achieve this by investigating the convex hulls of the sets of feasible solutions to the respective MIPs.

We shall define the *quadratic assignment polytope* as

$$\mathcal{QAP}_n := \operatorname{conv}\bigl(\{(x^C, y^{\mathcal{E}_n(C)}) \mid C \in \mathcal{CLQ}_n^n\}\bigr)$$

and the *symmetric quadratic assignment polytope* as

$$\mathcal{SQAP}_n := \operatorname{conv}\bigl(\{(x^C, z^{\mathcal{F}_n(C)}) \mid C \in \mathcal{CLQ}_n^n\}\bigr).$$

Before starting to consider the connection between these two polytopes, we want to mention the following facts.

OBSERVATION 1. *The two polytopes $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$ are invariant under permutations of the rows, permutations of the columns, and "transposition" of the node set $\mathcal{V}_n$. In particular, for each of the two polytopes, all the cones induced at the vertices are isomorphic.*

For the first one, the QAP-polytope, investigations were started by [24, 21, 14, 16]. There is not much known about the second one, the SQAP-polytope. Basically, there is only a conjecture of [24] and [21] concerning the dimension of $\mathcal{SQAP}_n$, which we will prove to be valid in Theorem 4.2.

This paper is concerned with the SQAP-polytope. However, it turns out that $\mathcal{SQAP}_n$ and $\mathcal{QAP}_n$ are closely related—although they are not isomorphic (e.g., we will see that they have different dimensions). The situation is quite similar to the relationship between the *asymmetric* and the *symmetric traveling salesman polytope.* While it is difficult to carry over results from the symmetric to the asymmetric case, this is (sometimes) possible for the opposite direction.

Next, we want to explain the relationship between the QAP- and the SQAP-polytope. Formally, the two polytopes are connected by

$$\mathcal{SQAP}_n = \mathrm{sym}_n(\mathcal{QAP}_n).$$

(Just consider the vertices to see this.)

We define an inequality (equation) $(u,v)^T(x,y) \le (=)\omega$ with $(u,v) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ and $\omega \in \mathbb{R}$ to be *symmetric* if and only if components of v that belong to mates are equal, i.e., $v_e = v_{\tau(e)}$ for all $e \in \mathcal{E}_n$. A face of $\mathcal{QAP}_n$ is called *symmetric* if there is a symmetric inequality defining that face. Even if a face of $\mathcal{QAP}_n$ is defined by a nonsymmetric inequality, it may be symmetric. This is because in general a face is defined by many different inequalities (even in the case of a facet, due to the low-dimensionality of $\mathcal{QAP}_n$), but in order to be symmetric it is required that there exists only one among these inequalities which is symmetric.

Let $(u,v)^T(x,y) \le (=)\omega$ be a symmetric valid inequality (equation) for the polytope $\mathcal{QAP}_n$. It induces a valid inequality (equation) $(u,w)^T(x,z) \le (=)\omega$ for $\mathcal{SQAP}_n$ with $w_{\mathrm{HYP}(e)} := v_e$ for all $e \in \mathcal{E}_n$. Conversely, every valid inequality (equation) for $\mathcal{SQAP}_n$ induces a symmetric valid inequality (equation) for $\mathcal{QAP}_n$. From this, we obtain the following.

Observation 2. *There is a one-to-one correspondence between the symmetric faces of $\mathcal{QAP}_n$ and the faces of $\mathcal{SQAP}_n$. If we identify the faces of $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$ with the node sets of the cliques corresponding to their vertices, then that correspondence is inclusion-preserving.*

This observation translates into the relationship between the face lattices of the QAP- and the SQAP-polytopes.

Theorem 3.1. *The face lattice of $\mathcal{SQAP}_n$ arises by restricting the face lattice of $\mathcal{QAP}_n$ to the symmetric faces. (Note that $\emptyset$ and $\mathcal{QAP}_n$ itself are symmetric faces of $\mathcal{QAP}_n$.)*

Corollary 3.2. *A symmetric proper face of $\mathcal{QAP}_n$ induces a facet of $\mathcal{SQAP}_n$ if and only if there are only nonsymmetric faces strictly between itself and $\mathcal{QAP}_n$ in the face lattice of $\mathcal{QAP}_n$.*

In general, it will be difficult to show that strictly between a certain symmetric face and the whole polytope there are only nonsymmetric faces of $\mathcal{QAP}_n$, because it is hard to prove that a set of faces is the complete set of faces containing a given face. However, in the special case that the face under consideration is a *ridge* of $\mathcal{QAP}_n$ (i.e., a face of two dimensions less than the whole polytope), the chances are better since it is a well-known fact that any ridge is the unique intersection of two facets.

Corollary 3.3. *If a symmetric ridge of $\mathcal{QAP}_n$ is the intersection of two nonsymmetric facets of $\mathcal{QAP}_n$, then it induces a facet of $\mathcal{SQAP}_n$.*

When investigating more closely the structure of a polytope defined as the convex hull of some points, one is very soon confronted with tasks such as computing the rank of a subset of these points or showing that such a subset spans a certain subspace. In both cases, one has to deal with linear combinations of the points, which one hopes to be sparse and to look somehow nice. Working with $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$, it turns out that such nice combinations are hard to obtain. This is mainly due to the facts that the coordinate vectors of the vertices look all the same up to certain permutations of the coordinates, and that there are no pairs among them having only slightly differing supports. On the other hand, for both of the polytopes a lot of equations are holding, indicating some redundancy in the problem definition. This motivated us to try to map the polytopes isomorphically into other spaces (of lower dimensions) in such a way that the coordinate vectors of the resulting vertices have nicer structures.

Let $\mathcal{A} \subset \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ be the affine subspace of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ defined by (2.1)–(2.4), i.e., $\mathcal{A} \subseteq \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ is the set of solutions to the equation system

$$\begin{aligned} x(\mathrm{row}_i^{(n)}) &= 1 \quad (i = 1, \dots, n), \\ x(\mathrm{col}_j^{(n)}) &= 1 \quad (j = 1, \dots, n), \\ -x_{(i,j)} + y((i,j) : \mathrm{row}_k^{(n)}) &= 0 \quad (i, j, k = 1, \dots, n, i \neq k), \\ -x_{(i,j)} + y((i,j) : \mathrm{col}_l^{(n)}) &= 0 \quad (i, j, l = 1, \dots, n, j \neq l), \end{aligned}$$

and let $\widehat{\mathcal{A}} \subset \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ be the affine subspace of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ defined by (2.7)–(2.10), i.e., $\widehat{\mathcal{A}} \subseteq \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ is the set of solutions to the system

$$\begin{aligned} x(\mathrm{row}_i^{(n)}) &= 1 \quad (i = 1, \dots, n), \\ x(\mathrm{col}_j^{(n)}) &= 1 \quad (j = 1, \dots, n), \\ -x_{(i,j)} - x_{(k,j)} + z(\Delta_{(i,j)}^{(k,j)}) &= 0 \quad (i, j, k = 1, \dots, n, i < k), \\ -x_{(i,j)} - x_{(i,l)} + z(\Delta_{(i,j)}^{(i,l)}) &= 0 \quad (i, j, l = 1, \dots, n, j < l). \end{aligned}$$

We will show that in both cases for the affine subspaces defined above all variables corresponding to vertices and edges, respectively, hyperedges involving the nth row or the nth column (the same holds for any row and any column) are redundant in the sense that the projections onto the linear subspaces of the original spaces obtained by setting all these variables to zero produce isomorphic images of these two affine subspaces. Since the two polytopes under consideration are contained in the respective affine subspaces, this implies that these projections yield isomorphic images of the polytopes.

Let $W := \mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)}$, $E := \{e \in \mathcal{E}_n \mid e \cap W \neq \emptyset\}$, and $F := \{f \in \mathcal{F}_n \mid f \cap W \neq \emptyset\}$. Define $\mathcal{U} := \{(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid x_W = 0, y_E = 0\}$ and $\widehat{\mathcal{U}} := \{(x, z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \mid x_W = 0, z_F = 0\}$. Let $\pi : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathcal{U}$ be the orthogonal projection onto $\mathcal{U}$, and $\widehat{\pi} : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \longrightarrow \widehat{\mathcal{U}}$ be the orthogonal projection onto $\widehat{\mathcal{U}}$.

PROPOSITION 3.4. *$\pi(\mathcal{A})$ is affinely isomorphic to $\mathcal{A}$ and $\widehat{\pi}(\widehat{\mathcal{A}})$ is affinely isomorphic to $\widehat{\mathcal{A}}$.*

Proof. We prove only the symmetric part of the proposition. The nonsymmetric part can be shown quite similarly [16].

First, we show that there is a way to express the components of points in $\widehat{\mathcal{A}}$ belonging to elements in W and F linearly by the components belonging to elements in $\mathcal{V}_n \setminus W$ and $\mathcal{F}_n \setminus F$.

The first observation is that this is possible for the elements in W using equations of the type $x(\mathrm{row}_i^{(n)}) = 1$ and $x(\mathrm{col}_j^{(n)}) = 1$. Now, we consider F. Here, it suffices to consider three possibilities for a hyperedge $\langle i, j, k, l \rangle \in F$. The first two are $i, j, k < n,\ l = n$ and $i, j, l < n,\ k = n$. Using $-x_{(i,j)} - x_{(k,j)} + z(\Delta_{(i,j)}^{(k,j)}) = 0$, respectively, $-x_{(i,j)} - x_{(i,l)} + z(\Delta_{(i,j)}^{(i,l)}) = 0$, the first two possibilities are done. The possibility remains that $i, j < n,\ k = n,\ l = n$. Here, we consider (e.g.) $-x_{(i,j)} - x_{(i,n)} + z(\Delta_{(i,j)}^{(i,n)}) = 0$, which allows to express $z_{\langle i,j,n,n \rangle}$ since we can already express $z_{\langle i,j,k,n \rangle}$ for $k < n$.

Up to now, we have shown that there is a linear function $\widehat{\psi} : \mathbb{R}^{\mathcal{V}_n \setminus W} \times \mathbb{R}^{\mathcal{F}_n \setminus F} \longrightarrow \mathbb{R}^W \times \mathbb{R}^F$ such that for all $(x, z) \in \widehat{\mathcal{A}}$ we have $(x_W, z_F) = \widehat{\psi}(x_{\mathcal{V}_n \setminus W}, z_{\mathcal{F}_n \setminus F})$. Hence

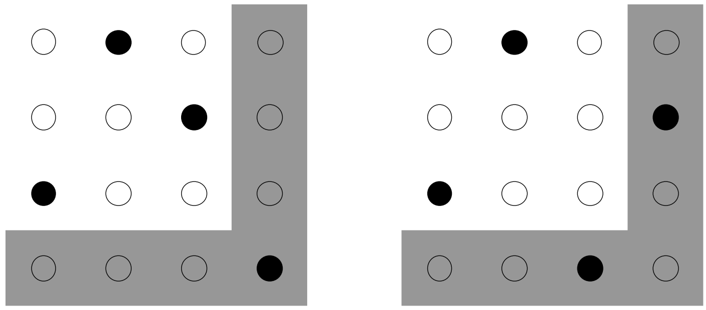

FIG. 3.1. *The effect of the projection.*

$\widehat{\phi} : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ defined via $\widehat{\phi}(x,z) = (x', z')$ with

$$(x'_W, z'_F) := (x_W, z_F) - \widehat{\psi}(x_{\mathcal{V}_n \setminus W}, z_{\mathcal{F}_n \setminus F}),$$
$$(x'_{\mathcal{V}_n \setminus W}, z'_{\mathcal{F}_n \setminus F}) := (x_{\mathcal{V}_n \setminus W}, z_{\mathcal{F}_n \setminus F})$$

is an affine transformation (note that the corresponding matrix is an upper triangular one having 1's everywhere on the main diagonal) of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ that induces on $\widehat{\mathcal{A}}$ the orthogonal projection onto $\widehat{\mathcal{U}}$. □

We identify the linear spaces $\mathcal{U}$ and $\widehat{\mathcal{U}}$ with the spaces $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$ and $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{F}_{n-1}}$, respectively. Hence,

$$\mathcal{QAP}^{\star}_{n-1} := \pi(\mathcal{QAP}_n) \subset \mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$$

is a polytope in $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$ that is isomorphic to $\mathcal{QAP}_n$, and

$$\mathcal{SQAP}^{\star}_{n-1} := \widehat{\pi}(\mathcal{SQAP}_n) \subset \mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{F}_{n-1}}$$

is a polytope in $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{F}_{n-1}}$ that is isomorphic to $\mathcal{SQAP}_n$.

Since the vertices of these two polytopes arise as the projections of the vertices of the two original polytopes, one obtains that they are the respective characteristic vectors of the $(n-1)$- and the $(n-2)$-cliques of $\mathcal{G}_{n-1}$ (cf. Figure 3.1).

We want to make the isomorphism between $\mathcal{QAP}_n$ and $\mathcal{QAP}^{\star}_{n-1}$ as well as the one between $\mathcal{SQAP}_n$ and $\mathcal{SQAP}^{\star}_{n-1}$ a little more explicit. We denote by $\kappa : \mathcal{CLQ}^n_n \longrightarrow \mathcal{CLQ}^{n-1}_{n-1} \cup \mathcal{CLQ}^{n-1}_{n-2}$ the map defined by removing from a given n-clique in $\mathcal{G}_n$ the node(s) in the nth row and in the nth column. Notice that κ is one to one.

REMARK 1. *If two faces of $\mathcal{QAP}_n$ and $\mathcal{QAP}^{\star}_{n-1}$, respectively, $\mathcal{SQAP}_n$ and $\mathcal{SQAP}^{\star}_{n-1}$, correspond to each other with respect to the isomorphism induced by π, respectively, $\widehat{\pi}$, then their vertices (identified with cliques) correspond to each other by the bijection κ.*

This remark describes the relationship between the faces from the "inner view," i.e., in terms of the vertices. Next, we want to describe the "outer relationship," i.e., the relationship between inequalities defining corresponding faces.

REMARK 2.

(i) *If a face of $\mathcal{QAP}_n$, respectively, $\mathcal{SQAP}_n$, is defined by an inequality that has zero coefficients for all elements in $W \cup E$, respectively, $W \cup F$, then an inequality defining the corresponding face of $\mathcal{QAP}^{\star}_{n-1}$, respectively, $\mathcal{SQAP}^{\star}_{n-1}$, is obtained by projecting the coefficient vector of that inequality via π, respectively, $\widehat{\pi}$. (Note that for every face of $\mathcal{QAP}_n$, respectively, $\mathcal{SQAP}_n$, there is a* *defining inequality having zero coefficients at W and E, respectively F. This is due to the fact that the columns of the equation system defining the affine subspace $\mathcal{A}$, respectively, $\widehat{\mathcal{A}}$, corresponding to $W \cup E$, respectively, $W \cup F$, are linearly independent, as shown in the proof of Proposition* 3.4.)

(ii) *From every inequality defining a face of $\mathcal{QAP}^\star_{n-1}$, respectively, $\mathcal{SQAP}^\star_{n-1}$, one obtains an inequality defining the corresponding face of $\mathcal{QAP}_n$, respectively, $\mathcal{SQAP}_n$, by zero-lifting.*

The following "star-analogons" to some facts observed for $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$ hold. First, also the "star-polytopes" are invariant under permutations of rows, permutations of columns, or "transposition" of the node set $\mathcal{V}_n$. Second, as in the relationship between $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$, by identifying mates any symmetric inequality (equation) for $\mathcal{QAP}^\star_n$ gives rise to an inequality (equation) for $\mathcal{SQAP}^\star_n$, and any inequality (equation) for $\mathcal{SQAP}^\star_n$ gives rise to a symmetric inequality (equation) for $\mathcal{QAP}^\star_n$.

THEOREM 3.5. *The face lattice of $\mathcal{SQAP}^\star_n$ arises by restricting the face lattice of $\mathcal{QAP}^\star_n$ to the symmetric faces.*

COROLLARY 3.6. *A symmetric proper face of $\mathcal{QAP}^\star_n$ induces a facet of $\mathcal{SQAP}^\star_n$ if and only if there are only nonsymmetric faces strictly between itself and $\mathcal{QAP}^\star_n$ in the face lattice of $\mathcal{QAP}^\star_n$.*

COROLLARY 3.7. *If a symmetric ridge of $\mathcal{QAP}^\star_n$ is the intersection of two nonsymmetric facets of $\mathcal{QAP}^\star_n$, then it induces a facet of $\mathcal{SQAP}^\star_n$.*

We close this section by the following "inductive construction" of $\mathcal{SQAP}_{n+1}$. It establishes a kind of "self-similarity" that shows another symmetry of the SQAP-polytope. The proof of the theorem can be found in [17].

THEOREM 3.8. *For $n \geq 1$ there are $n+1$ affine maps $\iota_\alpha : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_{n+1}} \times \mathbb{R}^{\mathcal{F}_{n+1}}$ ($\alpha = 0, \ldots, n$) such that for the $n+1$ images $\mathcal{Q}_\alpha := \iota_\alpha(\mathcal{SQAP}_n)$ ($\alpha = 0, \ldots, n$) of $\mathcal{SQAP}_n$ the following hold:*

(i) *Every $\mathcal{Q}_\alpha$ is isomorphic to $\mathcal{SQAP}_n$.*
(ii) *Each $\mathcal{Q}_\alpha$ is a face of $\mathcal{SQAP}_{n+1}$.*
(iii) *The $\mathcal{Q}_\alpha$ have pairwise empty intersection.*
(iv) $\mathcal{SQAP}_{n+1} = \operatorname{conv}\left(\bigcup_{\alpha=0}^{n} \mathcal{Q}_\alpha\right)$.

4. Dimension and trivial facets of $\mathcal{SQAP}_n$. In this section, we will present some basic results concerning the facial structure of the SQAP-polytope. First, we examine two sets of equations that will turn out to describe the affine hulls of $\mathcal{QAP}^\star_n$, respectively, $\mathcal{SQAP}^\star_n$. For this, we make another notational convention. For two disjoint subsets $S, T \subset \mathcal{V}_n$, $S \cap T = \emptyset$, we define $\langle S : T \rangle := \{\{v,w\} \cup \tau(\{v,w\}) \mid \{v,w\} \in (S : T)\}$. Remembering that the vertices of both $\mathcal{QAP}^\star_n$ and $\mathcal{SQAP}^\star_n$ correspond to the n- and $(n-1)$-cliques of $\mathcal{G}_n$, one verifies that

$$x(\text{row}_i^{(n)}) + x(\text{row}_k^{(n)}) - y(\text{row}_i^{(n)} : \text{row}_k^{(n)}) = 1 \qquad (i < k) \tag{4.1}$$

and

$$x(\text{col}_j^{(n)}) + x(\text{col}_l^{(n)}) - y(\text{col}_j^{(n)} : \text{col}_l^{(n)}) = 1 \qquad (j < l) \tag{4.2}$$

are valid for $\mathcal{QAP}^\star_n$, and

$$x(\text{row}_i^{(n)}) + x(\text{row}_k^{(n)}) - z(\langle\text{row}_i^{(n)} : \text{row}_k^{(n)}\rangle) = 1 \qquad (i < k) \tag{4.3}$$

and

$$x(\text{col}_j^{(n)}) + x(\text{col}_l^{(n)}) - z(\langle\text{col}_j^{(n)} : \text{col}_l^{(n)}\rangle) = 1 \qquad (j < l) \tag{4.4}$$

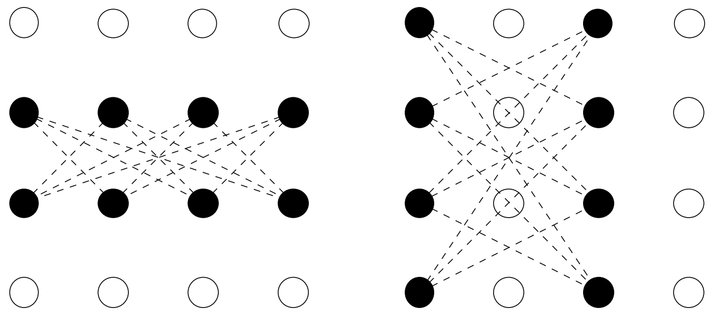

FIG. 4.1. *The equations* (4.1), (4.3) *and* (4.2), (4.4).

hold for $\mathcal{SQAP}_n^\star$ (cf. Figure 4.1).

We denote the system (4.1), (4.2) by $D(x, y) = d$ and the system (4.3), (4.4) by $\widehat{D}(x, z) = \widehat{d}$.

By saying that $\langle i, j, k, l \rangle$ $(i < k, j < l)$ is smaller than $\langle i', j', k', l' \rangle$ $(i' < k', j' < l')$ if and only if (i, k, j, l) is lexicographically smaller than (i', k', j', l'), we introduce an ordering on the hyperedges $\mathcal{F}_n$. After permutation of the columns with respect to this order the "z-part" of the matrix $\widehat{D}$ has the following shape ($n = 3$):

$$\begin{pmatrix} 1 & 1 & 1 & & & & & & \\ & & & 1 & 1 & 1 & & & \\ & & & & & & 1 & 1 & 1 \\ 1 & & & 1 & & & 1 & & \\ & 1 & & & 1 & & & 1 & \\ & & 1 & & & 1 & & & 1 \end{pmatrix}.$$

But this is the node-edge-incidence matrix of the complete bipartite graph on $\frac{n(n-1)}{2} + \frac{n(n-1)}{2}$ nodes, where the left shore corresponds to the (unordered) pairs of rows, and the right shore corresponds to the (unordered) pairs of columns of $\mathcal{V}_n$. The bases of the node-edge-incidence matrix of the complete bipartite graph on $m + m$ nodes are well known to correspond to the spanning trees of that graph [4]. Using the observation that $\widehat{D}$ does not have full row-rank (since, e.g., the sum of all equations in (4.3) equals the sum of all equations in (4.4)), this leads to the following characterization of all bases of $\widehat{D}$ that do not intersect the "x-part" of $\widehat{D}$.

PROPOSITION 4.1.

(i) *Precisely one (arbitrary) equation in* $\widehat{D}(x, z) = \widehat{d}$ *is redundant, in particular* $\operatorname{rank}(\widehat{D}) = n(n-1) - 1$.

(ii) *A subset* $B \subseteq \mathcal{F}_n$ *of hyperedges corresponds to a basis of* $\widehat{D}$ *if and only if*
   (a) $|B| = n(n-1) - 1$;
   (b) *there is no sequence* $(f_0, f_0', f_1, f_1', \ldots, f_{k-1}, f_{k-1}')$ $(k \geq 2)$ *of hyperedges in* B *such that* f_α *and* f_α' *connect the same rows of* $\mathcal{V}_n$ *and* f_α' *and* $f_{(\alpha+1) \bmod k}$ *connect the same columns of* $\mathcal{V}_n$ *for all* $\alpha = 0, \ldots, k-1$.

In [16] we showed that $D(x, y) = d$ is a complete equation system for $\mathcal{QAP}_n^\star$. But the system $D(x, y) = d$ consists only of symmetric equations. Hence, we can deduce that $\widehat{D}(x, z) = \widehat{d}$ must be a complete system of equations for $\mathcal{SQAP}_n^\star$, since the equations for $\mathcal{SQAP}_n^\star$ correspond precisely to the symmetric equations for $\mathcal{QAP}_n^\star$. (In fact, one can deduce the "completeness" of $\widehat{D}(x, z) = \widehat{d}$ also from the proof of Theorem 4.4.)

Consequently, the dimension of $\mathcal{SQAP}_n^\star$ is $n^2 + \frac{n^2(n-1)^2}{4} - (n(n-1)-1)$. By the isomorphism between $\mathcal{SQAP}_n$ and $\mathcal{SQAP}_{n-1}^\star$, one obtains the following theorem.

THEOREM 4.2.

$$\dim(\mathcal{SQAP}_n) = (n-1)^2 + \frac{(n-1)^2(n-2)^2}{4} - ((n-1)(n-2)-1).$$

[24] and [21] proved that the rank of the system (2.7)–(2.10) equals $(n-1)^2 + \frac{n^2(n-3)^2}{4}$ (which is equal to $(n-1)^2 + \frac{(n-1)^2(n-2)^2}{4} - ((n-1)(n-2)-1)$) and conjectured that this might be the dimension of $\mathcal{SQAP}_n$. Theorem 4.2 proves this conjecture. Moreover, knowing that the rank of this system equals $\dim(\mathcal{SQAP}_n)$, one can even conclude that the system (2.7)–(2.10) describes the affine hull of $\mathcal{SQAP}_n$. In addition, we want to give another simple proof that does not compute the rank of the system explicitly.

THEOREM 4.3.

$$\operatorname{aff}(\mathcal{SQAP}_n) = \{(x,z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \mid (x,z) \text{ satisfies } (2.7),\ldots,(2.10)\}.$$

Proof. It suffices to show that one can linearly combine the zero-liftings of (4.3) and (4.4) (for $n-1$) from (2.7)–(2.10) (for n), since then it is clear that the solution space of (2.7)–(2.10) for n—which is $\widehat{\mathcal{A}}$ (containing $\mathcal{SQAP}_n$)—is mapped isomorphically (cf. Proposition 3.4) by the projection $\widehat{\pi}$ into the solution space of (4.3), (4.4) for $n-1$, which we know from our considerations to have the same dimension as $\mathcal{SQAP}_n$.

Hence, by symmetry arguments, it suffices to exhibit a linear combination of (2.7)–(2.10) that yields

$$x(\text{row}_1^{(n)} \setminus \{(1,n)\}) + x(\text{row}_2^{(n)} \setminus \{(2,n)\}) - z(\langle \text{row}_1^{(n)} \setminus \{(1,n)\} : \text{row}_2^{(n)} \setminus \{(2,n)\}\rangle) = 1.$$

But this is obtained by adding $x(\text{row}_1^{(n)}) = 1$, $x(\text{row}_2^{(n)}) = 1$, $x_{(1,j)} + x_{(2,j)} - z(\Delta_{(1,j)}^{(2,j)}) = 0$ for all $1 \le j \le n-1$, and $-x_{(1,n)} - x_{(2,n)} + z(\Delta_{(1,n)}^{(2,n)}) = 0$, and finally dividing the resulting equation by 2. ∎

We just mention that the system (2.1)–(2.4) describes $\operatorname{aff}(\mathcal{QAP}_n)$ [24, 21, 16].

There is another nice gain when changing to the "star-polytopes." We pointed out in Corollary 3.2 that it is of interest to know that certain faces of the QAP-polytope are nonsymmetric. As mentioned above, this might not be directly seen, since a symmetric face of $\mathcal{QAP}_n$ can be defined by a nonsymmetric inequality. However, this is much easier for $\mathcal{QAP}_n^\star$.

OBSERVATION 3. *Due to the fact that all equations holding for $\mathcal{QAP}_n^\star$ are symmetric, in order to show that a given face of $\mathcal{QAP}_n^\star$ is nonsymmetric, it suffices to exhibit any nonsymmetric inequality defining it.*

For the nonsymmetric QAP-polytope, the nonnegativity constraints on y define facets, while $0 \le x \le 1$ and $y \le 1$ are already implied by $D(x,y) = d$ and $y \ge 0$ [24, 21, 16]. For the SQAP-polytope, the situation is a little bit different, as the following theorem shows.

THEOREM 4.4. *Let $n \ge 3$.*

(i) *The nonnegativity constraints $x \ge 0$ and $z \ge 0$ define facets of $\mathcal{SQAP}_n$.*

(ii) *The upper bounds $x \le 1$ and $z \le 1$ are implied by* (2.7)–(2.10) *and $x \ge 0$, $z \ge 0$.*

Proof. Part (ii) follows from the observation that (2.7) and (2.8) together with the nonnegativity of x imply $x \leq 1$. Furthermore, (2.7) and (2.8) even imply that the sum of any two x-variables that belong to the same row or column must be less than or equal to 1. Thus, from (2.9), (2.10), and the nonnegativity of z one obtains $z \leq 1$ as well.

To show part (i), it suffices to prove that $x \geq 0$ and $z \geq 0$ define facets of $\mathcal{SQAP}_n^\star$ (for all $n \geq 3$). We will show this only for $n \geq 5$, since this simplifies the proof. However, the claim is also true for $n = 3, 4$, as one may check by computer, for instance.

At this point, we introduce some techniques which we will also refer to in later proofs. Our usual way of proving that some inequality defines a facet of $\mathcal{SQAP}_n^\star$ is an indirect one. We denote by $L \subseteq \mathcal{CLQ}_n^n \cup \mathcal{CLQ}_{n-1}^n$ the set of cliques corresponding to the vertices of the considered face and by $\mathcal{L} := \{(x^C, z^{\mathcal{F}_n(C)}) - (x^{C'}, z^{\mathcal{F}_n(C')}) \mid C, C' \in L\}$ the set of all difference vectors of vertices of that face, i.e., $\text{lin}(\mathcal{L})$ is the subvectorspace belonging to the affine hull of the face. We choose a subset $B \subset \mathcal{F}_n$ that corresponds to a basis of the equation system $\widehat{D}(x, z) = \widehat{d}$ as well as one extra element $v_0 \in \mathcal{V}_n$ or $f_0 \in \mathcal{F}_n \setminus B$. Setting $\mathcal{B} := \{x^{v_0}\} \cup \{z^f \mid f \in B\}$, respectively, $\mathcal{B} := \{z^{f_0}\} \cup \{z^f \mid f \in B\}$, and providing that the face is a proper one, it remains to show that $\text{lin}(\mathcal{L} \cup \mathcal{B}) = \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$, since this implies that the dimension of $\text{lin}(\mathcal{L})$, which equals the dimension of the face, is at least $\dim(\mathcal{SQAP}_n^\star) - 1$. We show $\text{lin}(\mathcal{L} \cup \mathcal{B}) = \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ by successively combining the canonical unit vectors of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ from elements in $\mathcal{L} \cup \mathcal{B}$.

For constructing the necessary linear combinations, the following two lemmas are useful. For a subset $S \subseteq \mathcal{V}_n$ we denote by $\mathcal{H}_n/S = (\mathcal{V}_n/S, \mathcal{F}_n/S)$ the hypergraph obtained from $\mathcal{H}_n$ by deleting all nodes lying in a common row or column with a node in S and all hyperedges involving such nodes. Note that if S intersects the same number of rows as of columns, $\mathcal{H}_n/S$ is isomorphic to an $\mathcal{H}_k$ for some $k \leq n$.

LEMMA 4.5. *Let $C \in \mathcal{CLQ}_n^n$ be an n-clique and $v \in C$ a node in C such that $C, C \setminus \{v\} \in L$. Then we have*

$$x^v + z^{\langle v:C\setminus\{v\}\rangle} \in \text{lin}(\mathcal{L}).$$

Proof. This is due to $x^v + z^{\langle v:C\setminus\{v\}\rangle} = (x^C, z^{\mathcal{F}_n(C)}) - (x^{C\setminus\{v\}}, z^{\mathcal{F}_n(C\setminus\{v\})}) \in \text{lin}(\mathcal{L})$. □

LEMMA 4.6. *Let $1 \leq r, r_1, r_2 \leq n$ be pairwise distinct, and let $1 \leq c, c_1, c_2 \leq n$ be pairwise distinct. If there is an $(n-3)$-clique C in $\mathcal{H}_n/\{(r_1, c_1), (r, c), (r_2, c_2)\}$ such that*

$$\begin{aligned} (4.5) \quad & \{(r_1, c_1), (r, c), (r_2, c_2)\} \cup C, \quad \{(r_1, c_2), (r, c), (r_2, c_1)\} \cup C, \\ & \qquad \{(r_1, c_1), (r_2, c_2)\} \cup C, \quad \{(r_1, c_2), (r_2, c_1)\} \cup C \in L \end{aligned}$$

or

$$\begin{aligned} (4.6) \quad & \{(r_1, c), (r, c_2)\} \cup C, \quad \{(r, c_2), (r_2, c)\} \cup C, \\ & \qquad \{(r_2, c), (r, c_1)\} \cup C, \quad \{(r, c_1), (r_1, c)\} \cup C \in L, \end{aligned}$$

then

$$z^{\langle r_1,c_1,r,c\rangle} + z^{\langle r,c,r_2,c_2\rangle} - z^{\langle r_1,c_2,r,c\rangle} - z^{\langle r,c,r_2,c_1\rangle} \in \text{lin}(\mathcal{L})$$

(cf. Figure 4.2).

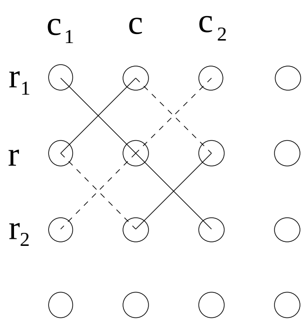

FIG. 4.2. *Notations of Lemma* 4.6.

Proof. In the first case, observe that

$$\begin{aligned} &z^{\langle r_1,c_1,r,c\rangle} + z^{\langle r,c,r_2,c_2\rangle} - z^{\langle r_1,c_2,r,c\rangle} - z^{\langle r,c,r_2,c_1\rangle} \\ =\, & z^{\{(r_1,c_1),(r,c),(r_2,c_2)\}\cup C} - z^{\{(r_1,c_1),(r_2,c_2)\}\cup C} \\ -\, & z^{\{(r_1,c_2),(r,c),(r_2,c_1)\}\cup C} + z^{\{(r_1,c_2),(r_2,c_1)\}\cup C} \in \operatorname{lin}(\mathcal{L}). \end{aligned}$$

For the second case, we have

$$\begin{aligned} &z^{\langle r_1,c_1,r,c\rangle} + z^{\langle r,c,r_2,c_2\rangle} - z^{\langle r_1,c_2,r,c\rangle} - z^{\langle r,c,r_2,c_1\rangle} \\ =\, & - z^{\{(r_1,c),(r,c_2)\}\cup C} + z^{\{(r,c_2),(r_2,c)\}\cup C} \\ & - z^{\{(r_2,c),(r,c_1)\}\cup C} + z^{\{(r,c_1),(r_1,c)\}\cup C} \in \operatorname{lin}(\mathcal{L}). \qquad \square \end{aligned}$$

Now, we proceed with the proof of Theorem 4.4. First, note that all trivial inequalities define proper faces of $\mathcal{SQAP}_n^\star$. To show that the nonnegativity constraints on x define facets of $\mathcal{SQAP}_n^\star$, it suffices to show this for $x_{(n,n)} \geq 0$. Hence, L consists of all n- and $(n-1)$-cliques of $\mathcal{H}_n$ that do not contain (n,n). We choose $B := \langle \mathrm{row}_1^{(n)} : \mathrm{row}_2^{(n)} \rangle \cup \langle \mathrm{col}_1^{(n)} : \mathrm{col}_2^{(n)} \rangle$ (cf. Proposition 4.1) and the extra element as $v_0 := (n,n)$.

Since in $\mathcal{H}_k$ there is always a k-clique not involving a prescribed node as long as $k \geq 2$, we can apply Lemma 4.6 for every choice of r, r_1, r_2, c, c_1, c_2. (Recall that we assume $n \geq 5$.) We combine all canonical unit vectors in $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ successively in five steps that are illustrated in Figure 4.3. For a number $a \in \{1,2\}$, we denote by $\overline{a}$ the number with $\{\overline{a}\} = \{1,2\} \setminus \{a\}$.

Step 1. $z^{\langle i,j,k,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for $i, j \in \{1,2\}$.

The case $k \in \{1,2\}$ or $l \in \{1,2\}$ is already clear by the choice of B. Hence, assume $k, l \notin \{1,2\}$. Choosing $r := i$, $r_1 := \overline{i}$, $r_2 := k$, $c := j$, $c_1 := \overline{j}$, and $c_2 := l$ Lemma 4.6 yields $z^{\langle \overline{i},\overline{j},i,j\rangle} + z^{\langle i,j,k,l\rangle} - z^{\langle \overline{i},l,i,j\rangle} - z^{\langle i,j,k,\overline{j}\rangle} \in \operatorname{lin}(\mathcal{L})$. Since all involved unit vectors but $z^{\langle i,j,k,l\rangle}$ are in $\mathcal{B}$, we are done.

Step 2. $z^{\langle i,j,k,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for $i \in \{1,2\}, j,k,l \geq 3$.

With $r := i$, $r_1 := \overline{i}$, $r_2 := k$, $c := j$, $c_1 := 1$, $c_2 := l$ one obtains from Lemma 4.6 that $z^{\langle \overline{i},1,i,j\rangle} + z^{\langle i,j,k,l\rangle} - z^{\langle \overline{i},l,i,j\rangle} - z^{\langle i,j,k,1\rangle} \in \operatorname{lin}(\mathcal{L})$. All involved unit vectors but $z^{\langle i,j,k,l\rangle}$ are either in $\mathcal{B}$ or already shown to be in $\operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ in Step 1.

Step 3. $z^{\langle i,j,k,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for $j \in \{1,2\}, i,k,l \geq 3$.

This is done analogously to Step 2.

Step 4. $z^{\langle i,j,k,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for $i,j,k,l \geq 3$.

This time, we choose $r := i$, $r_1 := 1$, $r_2 := k$, $c := j$, $c_1 := 1$, and $c_2 := l$. Lemma 4.6 gives $z^{\langle 1,1,i,j\rangle} + z^{\langle i,j,k,l\rangle} - z^{\langle 1,l,i,j\rangle} - z^{\langle i,j,k,1\rangle} \in \operatorname{lin}(\mathcal{L})$, which proves the

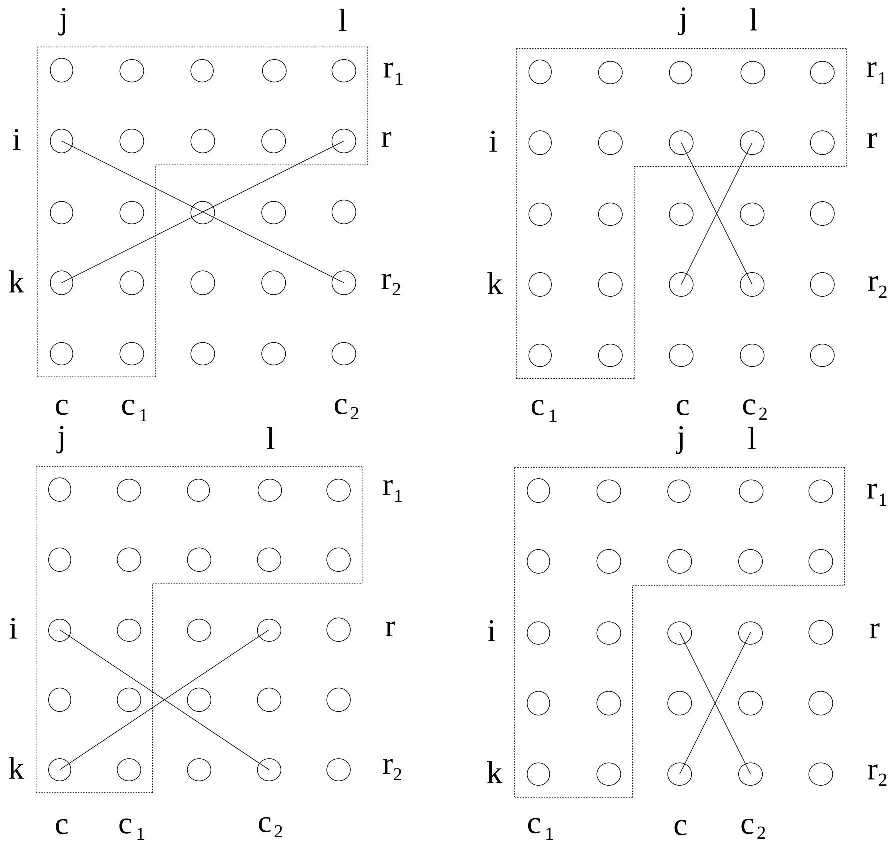

FIG. 4.3. *Examples for the hyperedges considered in Steps* 1–4 *of the proof of Theorem* 4.4. *The hyperedges inside the "angled box" are those forming the set B.*

claim, since all involved unit vectors but $z^{\langle i,j,k,l\rangle}$ are already shown to be in $\text{lin}(\mathcal{L}\cup\mathcal{B})$ in Steps 1, 2, or 3.

Step 5. $x^v \in \text{lin}(\mathcal{L}\cup\mathcal{B})$ for all $v \in \mathcal{V}_n$.

If $v = (n,n)$, we are done since $x^{(n,n)} \in \mathcal{B}$. So assume, $v \neq (n,n)$. Let $C \in \mathcal{CLQ}_n^n$ be any n-clique involving v but not (n,n). Using Lemma 4.5, we can combine x^v, since all unit vectors corresponding to hyperedges are already known to be in $\text{lin}(\mathcal{L}\cup\mathcal{B})$.

It remains to show that $z \geq 0$ define facets of $\mathcal{SQAP}_n^\star$. It suffices to show this for $z_{\langle n,n-1,n-1,n\rangle} \geq 0$. Now, L is the set of all n- and $(n-1)$-cliques of $\mathcal{H}_n$ that contain at most one node from $\{(n,n-1),(n-1,n),(n-1,n-1),(n,n)\}$. Note that it is always possible to find a k-clique in $\mathcal{H}_k$ that intersects $\{(k,k-1),(k-1,k),(k-1,k-1),(k,k)\}$ in at most one node as long as $k \geq 3$.

We choose B as above, and as the extra element, we take the hyperedge $\langle n,n-1,n-1,n\rangle$. Then, Steps 1, 2, and 3 work analogously. The only case in which Step 4 does not work is the case of the hyperedge $\langle n,n-1,n-1,n\rangle$, but this time this one is covered by the extra element. In Step 5, now we do not need an extra element anymore, since we can extend every node (also one from $\{(n,n-1),(n-1,n),(n-1,n-1),(n,n)\}$) to an n-clique not containing more than one node from $\{(n,n-1),(n-1,n),(n-1,n-1),(n,n)\}$. ☐

There is an alternative way of proving that the nonnegativity constraints $z \geq 0$

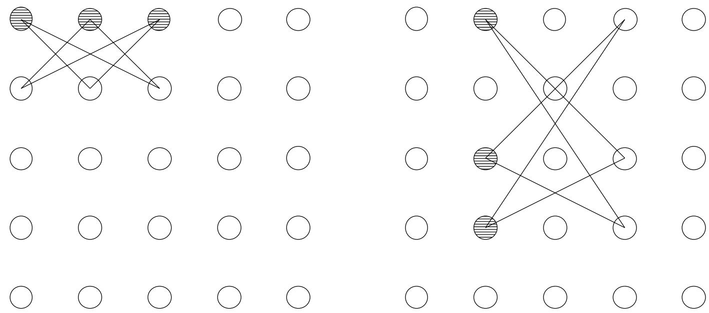

FIG. 5.1. *The curtain inequalities.*

define facets of $\mathcal{SQAP}_n^\star$. In [16] we showed that $y \geq 0$ define facets of $\mathcal{QAP}_n^\star$. By a slight modification of that proof, one can show that $y_e + y_{\tau(e)} \geq 0$ defines a ridge of $\mathcal{QAP}_n^\star$ for any edge $e \in \mathcal{E}_n$. Since that symmetric ridge is the intersection of the two nonsymmetric (cf. Observation 3) facets defined by $y_e \geq 0$ and $y_{\tau(e)} \geq 0$, the claim follows from Corollary 3.7.

5. The curtain facets. For any subset $S \subseteq \{1, \ldots, n\}$, we define for $i \in \{1, \ldots, n\}$ the *restriction of* $\text{row}_i^{(n)}$ *to* S as $\text{row}_i^{(n)}|_S := \{(i,j) \in \text{row}_i^{(n)} \mid j \in S\}$, and for $j \in \{1, \ldots, n\}$, we define $\text{col}_j^{(n)}|_S := \{(i,j) \in \text{col}_j^{(n)} \mid i \in S\}$ to be the *restriction of* $\text{col}_j^{(n)}$ *to* S.

One immediately verifies that the *row curtain inequalities*

$$-x(\text{row}_i^{(n)}|_S) + z(\langle \text{row}_i^{(n)}|_S : \text{row}_k^{(n)}|_S \rangle) \leq 0 \qquad (i \neq k, S \subseteq \{1, \ldots, n\}) \tag{5.1}$$

and the *column curtain inequalities*

$$-x(\text{col}_j^{(n)}|_S) + z(\langle \text{col}_j^{(n)}|_S : \text{col}_l^{(n)}|_S \rangle) \leq 0 \qquad (j \neq l, S \subseteq \{1, \ldots, n\}) \tag{5.2}$$

are valid for $\mathcal{SQAP}_n$ (cf. Figure 5.1).

These inequalities dominate the inequalities

$$-x(\text{row}_i^{(n)}|_S) + z(\langle (i,j) : \text{row}_k^{(n)}|_S \rangle) \leq 0 \qquad (i \neq k, S \subseteq \{1, \ldots, n\}, j \in S) \tag{5.3}$$

and

$$-x(\text{col}_j^{(n)}|_S) + z(\langle (i,j) : \text{col}_l^{(n)}|_S \rangle) \leq 0 \qquad (j \neq l, S \subseteq \{1, \ldots, n\}, i \in S) \tag{5.4}$$

proposed by [24] and [21].

The proof of the following theorem (which again uses the isomorphism between $\mathcal{SQAP}_n$ and $\mathcal{SQAP}_{n-1}^\star$) can be found in [17].

THEOREM 5.1. *All curtain inequalities with* $3 \leq |S| \leq n-3$ *define facets of* $\mathcal{SQAP}_n$.

We conclude this section with a consideration of the separation problem associated with the class of curtain inequalities. For this, let a (fractional) point $(\tilde{x}, \tilde{z}) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ be given. We want to find, e.g., a row curtain inequality using rows 1 and 2 (ordered) that "cuts off" the point $(\tilde{x}, \tilde{z})$. Hence, we want to find a subset $S \subseteq \{1, \ldots, n\}$ such that $-\tilde{x}(\text{row}_1^{(n)}|_S) + \tilde{z}(\langle \text{row}_1^{(n)}|_S : \text{row}_2^{(n)}|_S \rangle) > 0$. But this is exactly the task to find a characteristic vector ξ of $\{1,\dots,n\}$ that solves the *(unconstrained) Boolean quadratic* 0/1 *problem (BQP)*

$$\begin{array}{ll} \max & \displaystyle\sum_{j=1}^{n}\sum_{l=j+1}^{n}\alpha_{jl}\xi_j\xi_l+\sum_{j=1}^{n}\beta_j\xi_j \\ \text{subject to} & \xi\in\{0,1\}^n \end{array}$$

with $\alpha_{jl} := \tilde{z}_{\langle 1,j,2,l\rangle}$ and $\beta_j := -\tilde{x}_{(1,j)}$.

Hence, for each (ordered) pair of rows, respectively, columns, a BQP has to be solved. Although this is known to be $\mathcal{NP}$-hard in general, the special case of our separation problem, where all coefficients of quadratic terms are nonnegative, can be solved in polynomial time by computing a (directed) s-t minimum cut in a suitably defined graph (with nonnegative edge weights). This was first discovered by [23] (who formulated an algorithm in terms of flows) and further considered by [3] and other authors.

6. Lower bounds. For any instance of the QAP, the minimum the objective function achieves over the intersection of $\text{aff}(\mathcal{QAP}_n)$ and the nonnegative orthant is a lower bound for the optimal value of the respective QAP, called the *equation bound (EQB)*. This bound can be computed by solving the linear program arising from (2.1)–(2.4) and the nonnegativity constraints on the y-variables. Similarly, if the instance is symmetric, the minimum over the intersection of $\text{aff}(\mathcal{SQAP}_n)$ and the nonnegative orthant gives a lower bound, called the *symmetric equation bound (SEQB)*. This may be computed by solving the linear program defined by (2.7)–(2.10) and the nonnegativity constraints on x and z.

Let $(x,y)\in\text{aff}(\mathcal{QAP}_n)\cap(\mathbb{R}^{\mathcal{V}_n}_{\geq 0}\times\mathbb{R}^{\mathcal{E}_n}_{\geq 0})$ have value θ with respect to a symmetric objective function. Then $\text{sym}_n(x,y)\in\text{aff}(\mathcal{SQAP}_n)\cap(\mathbb{R}^{\mathcal{V}_n}_{\geq 0}\times\mathbb{R}^{\mathcal{F}_n}_{\geq 0})$ is a vector that also has value θ (with respect to the corresponding objective function for the symmetric formulation). Hence, SEQB can never be tighter than EQB.

It is possible to strengthen SEQB by the curtain inequalities. However, again one cannot obtain a lower bound that is tighter than EQB, since the curtain inequalities induce symmetric inequalities for the nonsymmetric problem that are already implied by the equations defining $\text{aff}(\mathcal{QAP}_n)$ and by the nonnegativity of the y-variables.

Hence, do the curtain inequalities have any computational value at all? Potentially, they do. By changing (in case of a symmetric instance) from the nonsymmetric problem formulation to the symmetric one, the number of variables is approximately divided by two. This leads to easier linear programs on the one hand, but to a potentially weaker bound SEQB on the other hand. So the question is, Can the curtain inequalities improve (empirically) the bound SEQB significantly toward EQB without losing too much of the efficiency gain made by the transition?

We want to mention at this point that EQB has turned out to be a very good lower bound for the QAP. The theoretical basis for this is a result due to [13] and [1] (extending work of [9]) which shows that EQB is always at least as good as the classical *Gilmore–Lawler Bound*, proposed independently by [10] and [20]. The practical indication for the quality of EQB was given most extensively in a computational study by [22]. They solved the linear programs that give the EQB for all instances in the quadratic assignment problem library (QAPLIB) [7] of size not exceeding $n = 30$ and found that EQB turned out to be the best-known lower bound in most cases.

Besides the more or less negligible weakening of the bound, there is one more important drawback when dealing with the symmetric instead of the nonsymmetric model. It has been observed and exploited by different people [1, 11] that the LP that has to be solved in order to compute EQB has a nice structure which allows one to design efficient heuristics to solve its dual, and thus to compute bounds that are nearly as good as EQB much faster than by evoking an LP-solver. Unfortunately, this nice structure is lost when changing to the symmetric model. However, once one starts to strengthen the bound by adding cutting planes (i.e., by exploiting polyhedral knowledge—see the remarks at the end of section 7), this structure is lost immediately, and the nonsymmetric model loses its advantage.

In order to investigate empirically the relative behavior of EQB, SEQB, and bounds obtained by adding some curtain inequalities to the (symmetric) formulation, we implemented a rudimentary cutting plane procedure for symmetric QAPs. This procedure initially solves the linear program that yields SEQB and afterward performs up to five cutting plane iterations with curtain inequalities. At each cutting plane iteration, we try to separate the current (fractional) solution by solving heuristically (i.e., repeating 100 times to guess a solution and improving it by a 2-opt procedure) a BQP for each ordered pair of rows/columns. If such a BQP ends with value greater than zero then we add the corresponding curtain inequality to the current linear program. This way, up to $2n(n-1)$ curtain inequalities may be added per iteration. It turned out that this naive (and fast) separation heuristic typically found many different violated inequalities per iteration.

The results show that in most cases, SEQB is not significantly worse than EQB. In fact, over all instances from the QAPLIB of sizes at most $n = 20$ the average ratio of SEQB and EQB is .986 (and the average ratio between the EQBs and the optimal solutions, known from the literature for all tested instances, is .859). Consequently, the curtain inequalities cannot improve SEQB very much. Usually, after five iterations the gap between SEQB and EQB is closed by about 30–40%. Regarding the quite small gaps between SEQB and EQB, the curtain inequalities do not seem to be computationally attractive. Therefore, we have not tried to improve the bounds by implementing an exact separation procedure for the curtain inequalities by the methods mentioned at the end of section 5.

The CPU times that are needed to compute SEQB are about three to four times smaller than the corresponding ones for EQB. They range from about 30 seconds for small ($n = 12$) instances up to about one hour for the hardest large ($n = 20$) instances. For more details on these experiments, we refer to [17].

7. Conclusion. We briefly discuss the context in which the work presented in this paper is located, in our opinion. Clearly, what we are finally concerned with is the exact (or at least provably good) solution of QAPs. The hope is that deeper polytopal knowledge of the problem will yield the necessary very good lower bounding procedures. Important steps that had already been performed were

- the evidence that EQB is empirically and theoretically a good lower bound,
- the basic polyhedral results on the QAP-polytope, and
- the definition of the SQAP-polytope.

The steps for the (quite natural) symmetric QAP that are done by the present paper are, from our point of view, the following.

- Our computational results indicate that changing the LP giving EQB in case of a symmetric instance in the natural way to a "symmetric LP" yielding SEQB does not decrease the quality of the lower bound significantly while accelerating the computations by a factor between three and four.
- It is useless to search for additional equations in order to improve the quality

of SEQB, since the used equation system is already complete.
- The curtain inequalities (strengthening inequalities proposed by [24] and [21]) seem to be computationally not very attractive (and they cannot be strengthened, since they already define facets).
- The methods presented in this paper, in particular the star-polytopes, provide possibilities for further investigations of the facial structure of the SQAP-polytope.

In particular, the last point in this list seems to be important. In fact, in the time between the submission of the first version and the preparation of the revised version of this paper, we have identified (using the techniques presented in this paper) a large class of facet-defining inequalities for the SQAP-polytope, the *box-inequalities*. They have turned out to be quite useful within cutting plane procedures for (symmetric) QAPs. Indeed, using these inequalities, it was for the first time possible to solve several instances from the QAPLIB to optimality by pure cutting plane algorithms, including three instances of size $n = 32$. We refer to [17, 15, 18] for details.

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