The QAP-Polytope and the Star-Transformation

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Polyhedral Combinatorics has been successfully applied to obtain considerable algorithmic progress towards the solution of many prominent hard combinatorial optimization problems. Until a few years ago, the quadratic assignment problem (QAP) was one of the exceptions. The work of Padberg and Rijal [15,13] has on the one hand yielded some basic facts about the associated quadratic assignment polytope, but has on the other hand shown that investigations even of the very basic questions (like the dimension, the affine hull, and the trivial facets) soon become extremely complicated. In this paper, we propose an isomorphic transformation of the "natural" realization of the quadratic assignment polytope, which simplifies the polyhedral investigations enormously. We demonstrate this by giving short proofs of the basic results on the polytope that indicate that, exploiting the techniques developed in this paper, deeper polyhedral investigations of the QAP now become possible.

Key words: Quadratic Assignment Problem, Polyhedral Combinatorics, QAP-Polytope (MSC Classification: 90C09, 90C10, 90C27)

1 Introduction

The methods of polyhedral combinatorics have yielded structural results as well as algorithms for the practical solution of many combinatorial optimization problems over the past thirty years. The most prominent examples among the $\mathcal{NP}$-hard problems include the traveling salesman problem, the maximum

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cut problem, and the linear ordering problem. If one compares this list with those $\mathcal{NP}$-hard combinatorial optimization problems that are usually considered "classical" one might miss the *quadratic assignment problem*, which we consider in the formulation

$$
\begin{array}{llll}
(\mathrm{QAP})^{(n)}_{c,d} & \min & \displaystyle\sum_{\substack{i,k=1\\ i<k}}^{n}\sum_{\substack{j,l=1\\ j\neq l}}^{n} d_{ijkl}x_{ij}x_{kl} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
 & \text{s.t.} & \displaystyle\sum_{j=1}^{n} x_{ij} = 1 & (i \in \{1,\dots,n\}) \\
 & & \displaystyle\sum_{i=1}^{n} x_{ij} = 1 & (j \in \{1,\dots,n\}) \\
 & & x_{ij} \in \{0,1\} & (i,j \in \{1,\dots,n\})
\end{array}
$$

of Lawler [11], who slightly generalized the original formulation of Koopmans and Beckmann [10]. And in fact, while there are numerous papers concerning polyhedral investigations of the other mentioned problems, one finds only a few occurrences of the quadratic assignment polytope.

Basically, this polytope was investigated only twice. First, it is treated in the work of Barvinok [4] as an example for the connection between the theory of representations of finite groups and combinatorial optimization polyhedra. Exploiting that deep theory Barvinok derives the dimension and some first facets of the quadratic assignment polytope. However, this method seems to apply only to these very basic questions. The second polyhedral investigation was carried out by Padberg and Rijal [15,13]. They derived basically the same results as Barvinok, using "classical" methods of polyhedral combinatorics. However, their treatment revealed that dealing with the quadratic assignment polytope, as it is defined naturally, leads to enormous technical difficulties of the following kind.

If one starts to investigate the structure of a polytope defined as the convex hull of some points, one is very soon confronted with tasks like computing the rank of a subset of these points or showing that such a subset spans a certain subspace. In both cases, one has to deal with linear combinations of the vertices of the polytope. Working with the natural realization of the quadratic assignment polytope, it turns out that such combinations with well-structured, sparse supports (i.e., sets of nonzero components) are hard to obtain, which is mainly due to the fact that there are no pairs among the vertices having only slightly different supports. In many cases, this makes it difficult to prove that certain subsets of vertices are affinely independent, which usually is much more convenient to do with sparse vectors for example.

In this paper, we describe how to overcome this "nastiness" by mapping the polytope isomorphically into a lower dimensional vector space, where the vertices allow some nice and simple linear combinations. This transformation seems to be crucial for the success of theoretical investigations of the quadratic assignment polytope. Without the simplifications of the proof-techniques that it yields, deeper results on the facial structure of the quadratic assignment polytope might be hard to obtain. We demonstrate the power of our transformation by deriving in a relatively simple way the dimension, the affine hull, and the trivial facets of the polytope.

The paper is organized as follows. In Section 2 we introduce a new way of formulating the quadratic assignment problem in graph theoretical terms and give the definition of the quadratic assignment polytope within this notational setting. Section 3 is the central part of the paper, where we develop the "star-transformation", and finally, we give short proofs for the dimension, the affine hull, and the trivial facets of the quadratic assignment polytope in Section 4.

Parts of the results presented in this paper can also be found in the (unpublished) technical report [9], which is a preliminary version of the present work.

2 The Polytope $\mathcal{QAP}_n$

2.1 Formulation as a Graph Problem

The set $\{1, \ldots, n\}$ will be used so frequently that it receives its own symbol. We will always denote $\mathcal{N} = \{1, \ldots, n\}$. Let $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ be the graph with node set $\mathcal{V}_n = \{(i,j) \mid i,j \in \mathcal{N}\}$ and edges

$$\mathcal{E}_n = \left\{ \{(i,j),(k,l)\} \in \binom{\mathcal{V}_n}{2} \middle| \, i \neq k, j \neq l \right\}$$

(where $\binom{\mathcal{V}_n}{2}$ is the set of all subsets of $\mathcal{V}_n$ having cardinality two). Figure 1 shows an example of such a graph. For ease of notation we define $[i,j,k,l] = \{(i,j),(k,l)\}$ for all edges $\{(i,j),(k,l)\} \in \mathcal{E}_n$. We call the subset $\text{row}_i^{(n)} = \{(i,j) \mid j \in \mathcal{N}\}$ the *i-th row* of $\mathcal{V}_n$ (for $i \in \mathcal{N}$) and the subset $\text{col}_j^{(n)} = \{(i,j) \mid i \in \mathcal{N}\}$ the *j-th column* of $\mathcal{V}_n$ (for $j \in \mathcal{N}$). If the context preserves from any ambiguity, then we usually omit the superscript and simply write row_i and col_j.

The connection between the graph $\mathcal{G}_n$ and the quadratic assignment problem comes from the fact that the maximum cliques of $\mathcal{G}_n$ are the n-cliques, and these correspond precisely to the $n \times n$-permutation matrices (see Figure 1).

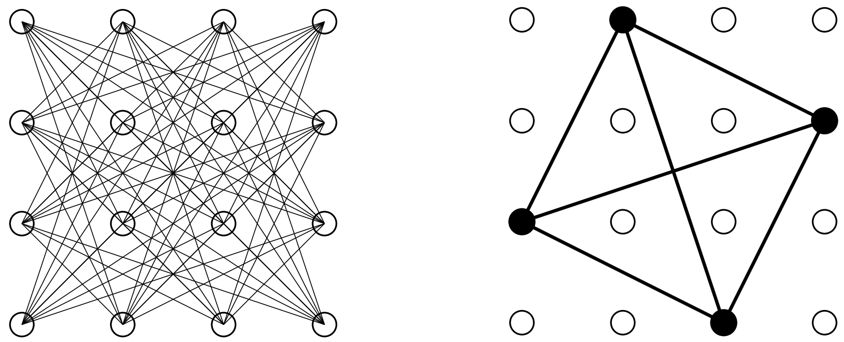


Fig. 1. The graph $\mathcal{G}_n$ and an example of an n-clique in it.

Hence, given an instance $(\mathrm{QAP})^{(n)}_{c,d}$ of the quadratic assignment problem, we weight the nodes and edges of $\mathcal{G}_n$ by $(c', d') \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$, where we set $c'_{(i,j)} = c_{ij}$ for each node $(i, j) \in \mathcal{V}_n$ and $d'_{[i,j,k,l]} = d_{ijkl}$ for each edge $[i, j, k, l] \in \mathcal{E}_n$. Then, solving $(\mathrm{QAP})^{(n)}_{c,d}$ means to find a minimally node- and edge-weighted n-clique in the graph $\mathcal{G}_n$ weighted by (c', d').

2.2 Definition and Elementary Properties of $\mathcal{QAP}_n$

Now we are ready to introduce the *quadratic assignment polytope.* We denote the characteristic vector of a subset $W \subseteq \mathcal{V}_n$ of nodes by $x^W \in \mathbb{R}^{\mathcal{V}_n}$ and the characteristic vector of a subset $F \subseteq \mathcal{E}_n$ of edges by y^F, i.e., we have

$$x^W_v = \begin{cases} 1 & \text{if } v \in W \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad x^F_e = \begin{cases} 1 & \text{if } f \in F \\ 0 & \text{otherwise.} \end{cases}$$

In particular, omitting the brackets for singletons, x^v (for $v \in \mathcal{V}_n$) and y^e (for $e \in \mathcal{E}_n$) are the unit vectors of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ that form the standard basis consisting of all 0/1-vectors with precisely one component equal to one.

For a subset $W \subseteq \mathcal{V}_n$ we denote by $\mathcal{E}_n(W) = \{\{v, w\} \in \mathcal{E}_n \mid v, w \in W\}$ the set of all edges having both nodes in W. The *incidence vector* of an n-clique $C \subset \mathcal{V}_n$ in $\mathcal{G}_n$ is the 0/1-vector $(x^C, y^C) = (x^C, y^{\mathcal{E}_n(C)})$. We define the *quadratic assignment polytope* to be the convex hull

$$\mathcal{QAP}_n = \operatorname{conv}\left\{(x^C, y^C) \mid C \text{ is an } n\text{-clique of } \mathcal{G}_n\right\}$$

of all incidence vectors of n-cliques of $\mathcal{G}_n$.

It can be shown that some well-known polytopes such as the *traveling salesman polytope* or the *linear ordering polytope* are certain projections of the quadratic assignment polytope. Furthermore, the quadratic assignment polytope is isomorphic to a face of the *boolean quadric polytope* (introduced by

Padberg [12]), which is itself isomorphic to the *cut polytope* on the complete graph (De Simone [5]). From this, for example, it can easily be deduced that the diameter of the quadratic assignment polytope equals one, since this holds for the cut polytope (Barahona and Mahjoub [3]).

An important property of the quadratic assignment polytope is the fact that it is invariant under permuting the rows or columns of the node set $\mathcal{V}_n$ and under "transposing" $\mathcal{V}_n$, which means that these operations induce *symmetries* of $\mathcal{QAP}_n$.

2.3 An Integer Linear Programming Formulation

For a vector $x \in \mathbb{R}^{\mathcal{V}_n}$ ($y \in \mathbb{R}^{\mathcal{E}_n}$) and for any subset $W \subseteq \mathcal{V}_n$ ($F \subseteq \mathcal{E}_n$) we denote by $x(W)$ ($y(F)$) the sum $\sum_{v \in W} x_v$ ($\sum_{e \in F} y_e$). For two disjoint subsets $S, T \subseteq \mathcal{V}_n$ the set of all edges in $\mathcal{E}_n$ with one endpoint in S and the other one in T is denoted by $(S : T)$. In case of singletons $S = \{s\}$ we omit the curly brackets.

Clearly, the equations

$$x(\mathrm{row}_i) = 1 \qquad (i \in \mathcal{N}) \tag{1}$$
$$x(\mathrm{col}_j) = 1 \qquad (j \in \mathcal{N}) \tag{2}$$
$$-x_{(i,j)} + y\left((i,j) : \mathrm{row}_k\right) = 0 \qquad (i, j, k \in \mathcal{N}, i \neq k) \tag{3}$$
$$-x_{(i,j)} + y\left((i,j) : \mathrm{col}_l\right) = 0 \qquad (i, j, l \in \mathcal{N}, j \neq l) \tag{4}$$

hold for all points in $\mathcal{QAP}_n$ (see Figure 2).

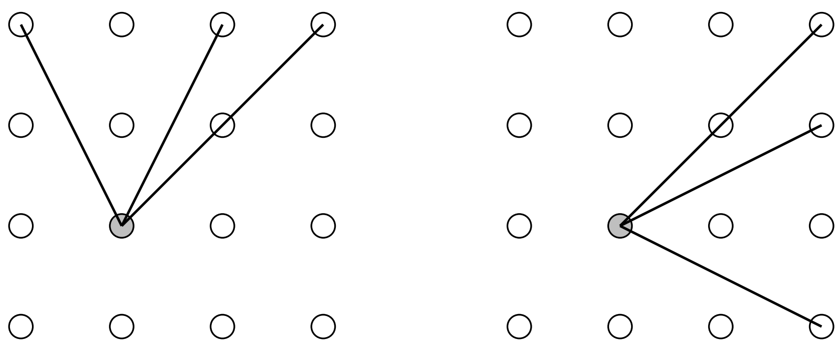


Fig. 2. The left-hand-side vectors of equations (3) and (4), where the solid lines indicate coefficients +1 and the grey dots indicate coefficients −1.

In fact, it was observed by several authors [8,6,15,13] that a vector $(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ is a vertex of $\mathcal{QAP}_n$ if and only if it satisfies (1), (2), (3), $y \geq 0$, and $x \in \{0,1\}^{\mathcal{V}_n}$. Moreover, Adams and Johnson [1] have proved that the lower bound which one can compute by solving the linear program arising from (1), (2), (3), (4), and the nonnegativity constraints $(x, y) \geq 0$ is always at least as good as the Gilmore/Lawler bound [7,11]. Extensive computational tests of Resende, Ramakrishnan, and Drezner [14] have shown that this bound is also very tight in practice.

3 A Different Representation: $\mathcal{QAP}^{\star}_{n^{\star}}$

3.1 An Isomorphic Projection of $\mathcal{QAP}_n$

The $2n+2n^2(n-1)$ many equations (1), ... ,(4) that are valid for the polytope $\mathcal{QAP}_n$ indicate some redundancy in the problem formulation. We will use this redundancy for finding another representation of the quadratic assignment polytope via a certain projection.

Let $\mathcal{A} \subset \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ be the affine subspace of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ defined by the equations (1), ... ,(4). We will show that the variables corresponding to vertices and edges involving the n-th row or the n-th column (the same holds for any row and any column) are redundant for $\mathcal{A}$ in the sense that the projection onto the linear subspace of the original space obtained by setting all these variables to zero produces an isomorphic image of this affine subspace. Since the polytope under consideration is contained in the affine subspace $\mathcal{A}$, this implies that the projection yields an isomorphic image of $\mathcal{QAP}_n$.

Let $W^{\star} = \mathrm{row}^{(n)}_n \cup \mathrm{col}^{(n)}_n$ and $F^{\star} = \{e \in \mathcal{E}_n \mid e \cap W^{\star} \neq \emptyset\}$. Define $\mathcal{U} = \{(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid x_{W^{\star}} = 0, y_{F^{\star}} = 0\}$, and let $\pi^{(n)} : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathcal{U}$ be the orthogonal projection onto $\mathcal{U}$.

Proposition 1 *The restriction of the projection $\pi^{(n)}$ to the affine subspace $\mathcal{A}$ of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ is a one-to-one map.*

PROOF. We will first show that there is a way to express the components of points in $\mathcal{A}$ belonging to elements in $W^{\star}$ and $F^{\star}$ linearly by the components belonging to elements in $\mathcal{V}_n \setminus W^{\star}$ and $\mathcal{E}_n \setminus F^{\star}$.

This is possible for the elements in $W^{\star}$ using the equations (1) and (2). In order to show the claim for $F^{\star}$, it suffices to consider three possibilities for an edge $[i, j, k, l] \in F$. The first two are $i, j, k < n$, $l = n$ and $i, j, l < n$, $k = n$. Using the suitable equation from (3) (with i, j, k in the first case) and (4) (with i, j, l in the second case), these two possibilities are done. It remains the possibility that $i, j < n$, $k = n$, $l = n$. We exploit (3) for i, j, n, which allows to express $y_{[i,j,n,n]}$ since we can already express $y_{[i,j,n,l]}$ for $l < n$.

Up to now, we have shown that there is a linear function $\psi : \mathbb{R}^{\mathcal{V}_n \setminus W^\star} \times \mathbb{R}^{\mathcal{E}_n \setminus F^\star} \longrightarrow \mathbb{R}^{W^\star} \times \mathbb{R}^{F^\star}$ such that for all $(x, y) \in \mathcal{A}$ we have $(x_{W^\star}, y_{F^\star}) = \psi(x_{\mathcal{V}_n \setminus W^\star}, y_{\mathcal{E}_n \setminus F^\star})$. Hence $\phi : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ defined via $\phi(x, y) = (x', y')$ with

$$\begin{aligned}(x'_{W^\star}, y'_{F^\star}) &= (x_{W^\star}, y_{F^\star}) - \psi(x_{\mathcal{V}_n \setminus W^\star}, y_{\mathcal{E}_n \setminus F^\star}),\\ (x'_{\mathcal{V}_n \setminus W^\star}, y'_{\mathcal{E}_n \setminus F^\star}) &= (x_{\mathcal{V}_n \setminus W^\star}, y_{\mathcal{E}_n \setminus F^\star})\end{aligned}$$

is an affine transformation (since the corresponding matrix is a triangular one having ones everywhere on the main diagonal) of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ that induces on $\mathcal{A}$ the orthogonal projection onto $\mathcal{U}$. □

We identify the linear space $\mathcal{U}$ with the space $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$. Hence, for $n^\star = n - 1$

$$\mathcal{QAP}^\star_{n^\star} = \pi^{(n)}(\mathcal{QAP}_n) \subset \mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}$$

is a polytope in $\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}$ that is isomorphic to $\mathcal{QAP}_n$.

Since the vertices of this polytope arise from the projections of the vertices of the original polytope "forgetting" the last row and the last column of $\mathcal{G}_n$, one obtains that they are the incidence vectors of the $n^\star$- and the $(n^\star - 1)$-cliques of $\mathcal{G}_{n^\star}$ (see Figure 3). Thus, by adapting the notations for the incidence vectors to $(n^\star - 1)$-cliques of $\mathcal{G}_{n^\star}$, we have

$$\mathcal{QAP}^\star_{n^\star} = \text{conv}\left\{(x^{C^\star}, y^{C^\star}) \mid C^\star \text{ is an } n^\star\text{- or an } (n^\star - 1)\text{-clique of } \mathcal{G}_{n^\star}\right\}.$$

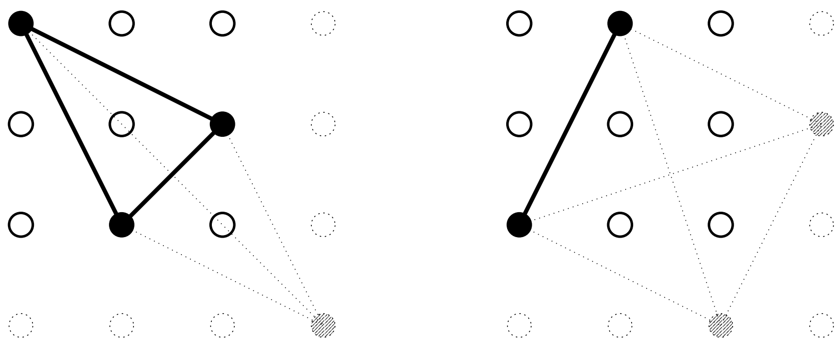


Fig. 3. The effect of the projection $\pi^{(n)}$.

We want to make the isomorphism that $\pi^{(n)}$ induces between $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_{n^\star}$, as well as between the corresponding face lattices a little more explicit. Denote by κ the map that assigns to every n-clique $C \subset \mathcal{V}_n$ of $\mathcal{G}_n$ the $n^\star$- or $(n^\star - 1)$-clique $C^\star \subset \mathcal{V}_{n^\star}$ of $\mathcal{G}_{n^\star}$ that arises from C by removing the node(s) in the n-th row and in the n-th column. Notice that κ is one-to-one.

Remark 2 *If two faces of $\mathcal{Q}AP_n$ and $\mathcal{Q}AP^\star_{n^\star}$ correspond to each other with respect to the isomorphism induced by $\pi^{(n)}$, then their vertices (identified with cliques) correspond to each other by the bijection κ.*

This remark describes the relationship between the faces from the "inner view", i.e., in terms of the vertices. Next, we want to describe the "outer relationship", i.e., the relationship between inequalities defining corresponding faces.

Remark 3

(i) If a face of $\mathcal{Q}AP_n$ is defined by an inequality that has zero-coefficients for all elements in $W^\star \cup F^\star$, then an inequality defining the corresponding face of $\mathcal{Q}AP^\star_{n^\star}$ is obtained by projecting the coefficient vector of that inequality via $\pi^{(n)}$. In fact, for every face of $\mathcal{Q}AP_n$ there is a defining inequality that has zero coefficients at $W^\star$ and $F^\star$, since the columns corresponding to $W^\star \cup F^\star$ of the equation system defining the affine subspace $\mathcal{A}$ are linearly independent, as shown in the proof of Proposition 1.

(ii) From every inequality defining a face of $\mathcal{Q}AP^\star_{n^\star}$ one obtains an inequality defining the corresponding face of $\mathcal{Q}AP_n$ by "zero-lifting", i.e., choosing zero as coefficient for every variable corresponding to $\mathcal{V}_n \setminus \mathcal{V}_{n^\star}$ or $\mathcal{E}_n \setminus \mathcal{E}_{n^\star}$.

As for $\mathcal{Q}AP_n$ (see Section 2.2), permuting the rows or columns as well as transposing the node set yields symmetries of the polytope $\mathcal{Q}AP^\star_{n^\star}$, i.e., it suffices also for $\mathcal{Q}AP^\star_{n^\star}$ to prove all results up to permutations of the rows or the columns as well as transposition of the node set.

3.2 A System of Equations

By mapping the polytope $\mathcal{Q}AP_n \subset \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ isomorphically (in particular, not changing its dimension) into the lower-dimensional space $\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}} = \mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$, we have reduced the dimensional gap between the polytope and the space it is located in. It would have been the best to make that gap even vanish, i.e., to obtain a full-dimensional representation of the quadratic assignment polytope. However, this is not reached by the projection $\pi^{(n)}$, as one sees from the equations coming up next.

Ending up with a full-dimensional polytope would be nice with respect to such goals like the uniqueness of facet-defining inequalities and clearly, for every low-dimensional polytope there is a possibility to map it isomorphically into another space where it is full-dimensional. However, the fact that the vertices of the representation of the quadratic assignment polytope have some nice combinatorial structure, as they do in the case of $\mathcal{Q}AP^\star_{n^\star}$, is of extreme importance to us. It seems that a full-dimensional representation of the quadratic assignment polytope satisfying this requirement is not possible.

We shall now exhibit the equations that still hold for $\mathcal{Q}AP^{\star}_{n^{\star}}$. Since every $n^{\star}$- or $(n^{\star}-1)$-clique of $\mathcal{G}_{n^{\star}}$ has an empty intersection with at most one row and with at most one column of $\mathcal{V}_{n^{\star}}$, the equations (where $\mathcal{N}^{\star}=\{1,\ldots,n^{\star}\}$)

$$x(\mathrm{row}_i \cup \mathrm{row}_k) - y(\mathrm{row}_i : \mathrm{row}_k) = 1 \qquad (i,k \in \mathcal{N}^{\star}, i<k) \qquad (5)$$
$$x(\mathrm{col}_j \cup \mathrm{col}_l) - y(\mathrm{col}_j : \mathrm{col}_l) = 1 \qquad (j,l \in \mathcal{N}^{\star}, j<l) \qquad (6)$$

are valid for $\mathcal{Q}AP^{\star}_{n^{\star}}$ (see Figure 4). Theorem 9 will show that (5) and (6) form a complete system of equations for $\mathcal{Q}AP^{\star}_{n^{\star}}$, i.e., the solution space of these equations is the affine hull of $\mathcal{Q}AP^{\star}_{n^{\star}}$.

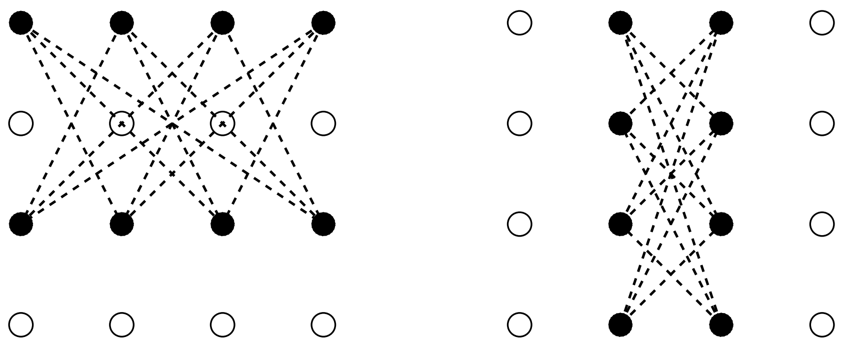


Fig. 4. The left-hand-side vectors of the equations (5) and (6), where the dashed lines indicate coefficients -1, and the filled dots indicate coefficients $+1$.

Let us investigate the system $D(x,y)=d$ of the equations (5), (6) more closely. A first observation is that this system has not full row rank, since summing up all equations (5) yields the same as summing up all equations (6). Hence, the rank of these $n^{\star}(n^{\star}-1)$ equations is at most $n^{\star}(n^{\star}-1)-1$.

We define a (total) ordering of the edges $\mathcal{E}_{n^{\star}}$ by requiring that each edge $[i,j,k,l] \in \mathcal{E}_{n^{\star}}$ with $i<k$ and $j<l$ has as its successor the edge $[i,l,k,j]$, and by ordering the edges $\{[i,j,k,l] \in \mathcal{E}_{n^{\star}} \mid i<k, j<l\}$ lexicographically according to the quadruples (i,k,j,l). After permuting the columns of D (that correspond to the edges of $\mathcal{G}_{n^{\star}}$) with respect to this ordering of $\mathcal{E}_{n^{\star}}$, these columns of D form the following $n^{\star}(n^{\star}-1) \times |\mathcal{E}_{n^{\star}}|$ matrix (for $n^{\star}=3$):

$$\begin{pmatrix}
1&1&1&1&1&1& & & & & & & & & & & & \\
 & & & & & &1&1&1&1&1&1& & & & & & \\
 & & & & & & & & & & & &1&1&1&1&1&1\\
1&1& & & & &1&1& & & & &1&1& & & & \\
 & &1&1& & & & &1&1& & & & &1&1& & \\
 & & & &1&1& & & & &1&1& & & & &1&1
\end{pmatrix}$$

The first three rows yield the equations (5) with $(i,k) = (1,2),(1,3),(2,3)$, while the others yield the equations (6) with $(j,l) = (1,2),(1,3),(2,3)$. Notice that a variable $y_{[i',j',k',l']}$ has a coefficient 1 precisely for $(i,k) = (i',k')$ and for $(j,l) = (j',l')$.

We are interested in the *bases of the matrix D* (also called the *bases of the equation system $D(x,y) = d$*), i.e., the maximal subsets of linearly independent columns of D. Since columns corresponding to edges $[i,j,k,l]$ and $[i,l,k,j]$ are identical, we can identify them for our considerations. But then, the resulting $n^\star(n^\star-1) \times \frac{1}{2}|\mathcal{E}_{n^\star}|$ matrix is the node-edge incidence matrix of the complete bipartite graph on $\frac{n^\star(n^\star-1)}{2}$ plus $\frac{n^\star(n^\star-1)}{2}$ nodes, where the left shore corresponds to the (unordered) pairs of rows, and the right shore corresponds to the (unordered) pairs of columns of $\mathcal{V}_{n^\star}$. Calling a pair $\{[i,j,k,l],[i,l,k,j]\}$ of edges of $\mathcal{G}_{n^\star}$ a pair of *mates*, we obtain a one-to-one correspondence between the edges in that complete bipartite graph and the pairs of mates.

The bases of the node-edge incidence matrix of the complete bipartite graph on N plus N nodes are well-known to correspond to the spanning trees of that graph (Balinski and Russakoff [2]). This leads to the following characterization of all bases of $D(x,y) = d$ that do not contain columns corresponding to nodes of $\mathcal{G}_{n^\star}$.

Proposition 4 *Let $n^\star \geq 2$.*

(i) Precisely one (arbitrary) equation in the system (5), (6) is redundant, in particular, the rank of this system is $n^\star(n^\star-1)-1$.

(ii) A subset $B \subset \mathcal{E}_{n^\star}$ of edges of $\mathcal{G}_{n^\star}$ corresponds to a basis of that system if and only if

(a) $|B| = n^\star(n^\star-1)-1$

(b) There is no pair of mates contained in B.

(c) There is no sequence $(e_0, e'_0, e_1, e'_1, \ldots, e_{r-1}, e'_{r-1})$ (with $r \geq 2$) of edges in B such that e_ρ and e'_ρ connect the same rows of $\mathcal{V}_{n^\star}$ and e'_ρ and $e_{(\rho+1) \bmod r}$ connect the same columns of $\mathcal{V}_{n^\star}$ for all $\rho = 0,\ldots,r-1$.

PROOF. Part (ii) follows from the discussion of the connection to the complete bipartite graph on $\frac{n(n-1)}{2}$ plus $\frac{n(n-1)}{2}$ nodes, and Part (i) follows from (ii) and the observation made above that the rank of $D(x,y) = d$ is at most $n^\star(n^\star-1)-1$. □

Later, when we prove results about the dimension of $\mathcal{QAP}^\star_{n^\star}$ or of one of its faces, we will always use one special basis of the equations system (5), (6) that we exhibit now. It is illustrated in Figure 5.

Corollary 5 *The columns corresponding to the set*

$$E_{\text{bas}}^{(n^\star)} = \{[1, j, 2, l] \in \mathcal{E}_{n^\star} \mid j < l\} \cup \{[i, 1, k, 2] \in \mathcal{E}_{n^\star} \mid i < k\}$$

form a basis of the equation system (5), (6).

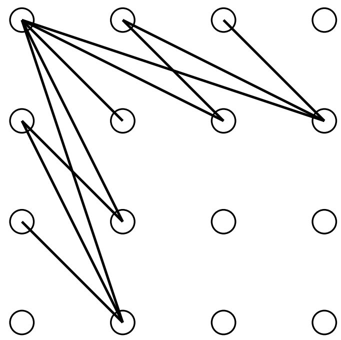


Fig. 5. The edges corresponding to the basis $E_{\text{bas}}^{(n^\star)}$.

3.3 A Proof Technique

The technique we will use to establish the dimension of $\mathcal{QAP}^\star_{n^\star}$ as well as in the proofs showing that a given inequality defines a facet of $\mathcal{QAP}^\star_{n^\star}$ is a variant of the "indirect method". We give an outline of this technique here.

First, we explain the technique for the dimension proof. Let L be the set of all $n^\star$- and $(n^\star - 1)$-cliques $C \subset \mathcal{V}_{n^\star}$, and let

$$\Delta_L = \left\{ (x^{C_1}, y^{C_1}) - (x^{C_2}, y^{C_2}) \mid C_1, C_2 \in L \right\}$$

be the set of all difference vectors of the incidence vectors of these cliques, i.e., the set of all differences of vertices of $\mathcal{QAP}^\star_{n^\star}$. Hence, Δ_L spans the linear subspace belonging to the affine subspace $\text{aff}(\mathcal{QAP}^\star_{n^\star})$. Denote the rank of the equation system (5), (6) by rank_{eq}. In order to prove that the equations (5), (6) completely describe the affine hull $\text{aff}(\mathcal{QAP}^\star_{n^\star})$ of $\mathcal{QAP}^\star_{n^\star}$ we have to show that $\dim(\text{aff}(\mathcal{QAP}^\star_{n^\star}))$ is at least (in fact, it will be equal to) $\dim(\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}) - \text{rank}_{\text{eq}}$. We will do this by showing that the linear dimension of Δ_L (which equals $\dim(\text{aff}(\mathcal{QAP}^\star_{n^\star}))$) is at least $\dim(\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}) - \text{rank}_{\text{eq}}$.

Let B be a set of edges belonging to a basis of the equation system (5), (6), in particular we have $|B| = \text{rank}_{\text{eq}}$. Clearly, one could also use a basis containing columns that belong to nodes, too. But we will always choose $B = E_{\text{bas}}^{(n^\star)}$ as in Corollary 5, and thus restrict our notations to the case that B contains no node. We denote by $\mathcal{B} = \{y^e \mid e \in B\}$ the set of all unit vectors belonging to B. Now it suffices to show

$$\text{lin}(\Delta_L \cup \mathcal{B}) = \mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}},$$

since by $\dim(\mathcal{B}) = \mathrm{rank}_{\mathrm{eq}}$ this implies $\dim(\mathrm{lin}(\Delta_L)) \geq \dim(\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}) - \mathrm{rank}_{\mathrm{eq}}$. We do this by showing that every vector in the basis $\{x^v \mid v \in \mathcal{V}_{n^\star}\} \cup \{y^e \mid e \in \mathcal{E}_{n^\star}\}$ of the vector space $\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}$ can be obtained as a linear combination of vectors in Δ_L and $\mathcal{B}$. In order to abbreviate the notations, we say that an edge or a node has been *combined* once the corresponding unit vector has been linearly combined. Hence, our goal is to combine all nodes and edges of $\mathcal{G}_{n^\star}$.

If we want to prove that a *proper* face $\mathcal{F}$ is a facet of $\mathcal{QAP}^\star_{n^\star}$, then we start with the set L containing not all $n^\star$- and $(n^\star - 1)$-cliques of $\mathcal{G}_{n^\star}$ but only those that belong to vertices of $\mathcal{F}$. Since we do not want to prove that the dimension of $\mathcal{F}$ equals that of $\mathcal{QAP}^\star_{n^\star}$ but $\dim(\mathcal{QAP}^\star_{n^\star}) - 1$, we enlarge the set $\mathcal{B}$ by one unit vector belonging either to a node v_0 or to an edge e_0, called the *extra element*, to a set $\mathcal{B}_0$. Proceeding as above with the "combination" of all unit vectors in the basis $\{x^v \mid v \in \mathcal{V}_{n^\star}\} \cup \{y^e \mid e \in \mathcal{E}_{n^\star}\}$ of $\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}$, starting from the modified set $\Delta_L \cup \mathcal{B}_0$, it will be proved that $\mathcal{F}$ is a facet of $\mathcal{QAP}^\star_{n^\star}$ as soon as all nodes and edges are combined (notice that $\mathcal{F}$ was supposed to be not the whole polytope).

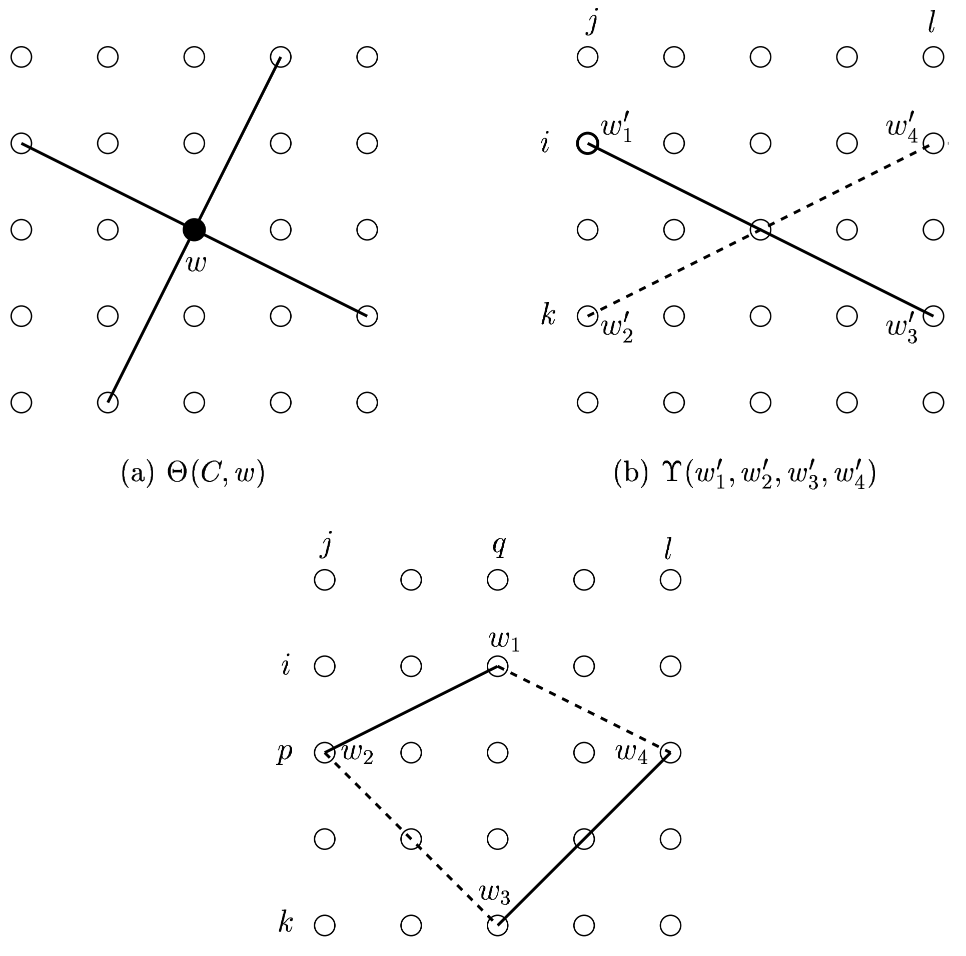
Proving this way that a given proper face of $\mathcal{QAP}^\star_{n^\star}$ defines a facet even contains a proof that (5) and (6) form a complete equation system for $\mathcal{QAP}^\star_{n^\star}$. We will use this fact and give in Section 4.1 one proof for both the dimension of $\mathcal{QAP}^\star_{n^\star}$ as well as for the fact that the nonnegativity constraints on the edge variables define facets of it.

3.4 Some Useful Vectors

The first convenient gain that we took from the transition to the "star-polytope" $\mathcal{QAP}^\star_{n^\star}$ was the equation system (5), (6) (that is not yet proved to be a complete one for $\mathcal{QAP}^\star_{n^\star}$, but will be soon in Section 4.1) with its structural connection to the node-edge incidence matrix of the complete bipartite graph. Now we will show that $\mathcal{QAP}^\star_{n^\star}$ allows to construct very simple vectors as linear combinations of its vertices.

Let $i, k, p \in \mathcal{N}^\star$ be three pairwise distinct row numbers of $\mathcal{V}_{n^\star}$, and let $j, l, q \in \mathcal{N}^\star$ be three pairwise distinct column numbers of $\mathcal{V}_{n^\star}$. The following vectors, where $w_1 = (i, q)$, $w_2 = (p, j)$, $w_3 = (k, q)$, $w_4 = (p, l)$, $w'_1 = (i, j)$, $w'_2 = (k, j)$, $w'_3 = (k, l)$, $w'_4 = (i, l)$, and $C \subset \mathcal{V}_{n^\star}$ is an $n^\star$-clique of $\mathcal{G}_{n^\star}$ containing the node $w \in C$, will be the most important auxiliaries for the combination of nodes and edges as explained in Section 3.3. They are illustrated in Figure 6.

$$\Theta(C, w) = x^w + \sum_{w' \in C \setminus w} y^{\{w,w'\}}$$
$$\Upsilon(w'_1, w'_2, w'_3, w'_4) = y^{[i,j,k,l]} - y^{[i,l,k,j]}$$
$$\Phi(w_1, w_2, w_3, w_4) = y^{[i,q,p,j]} - y^{[p,j,k,q]} + y^{[k,q,p,l]} - y^{[p,l,i,q]}$$



(a) $\Theta(C, w)$ (b) $\Upsilon(w'_1, w'_2, w'_3, w'_4)$

(c) $\Phi(w_1, w_2, w_3, w_4)$

Fig. 6. The three types of vectors provided by Lemmas 6, ... , 8

The following three lemmas give sufficient conditions for a set L of $n^\star$- and $(n^\star - 1)$-cliques of $\mathcal{G}_{n^\star}$ that guarantee these vectors to be members in $\text{lin}(\Delta_L)$, where Δ_L is, again, the set of all difference vectors of the incidence vectors of the cliques in L. We make one more notational convention for stating these lemmas. Let $W \subset \mathcal{V}_{n^\star}$ be a subset of nodes. We denote by $\mathcal{G}_{n^\star}/W$ the subgraph of $\mathcal{G}_{n^\star}$ that is induced by all rows and columns that do not intersect W. If W

intersects the same number of rows as of columns, then $\mathcal{G}_{n^\star}/W$ is isomorphic to some $\mathcal{G}_{\dot{n}}$ with $\dot{n} \le n^\star$.

Lemma 6 *Let L be a set of $n^\star$- and $(n^\star-1)$-cliques of $\mathcal{G}_{n^\star}$. If for an $n^\star$-clique C of $\mathcal{G}_{n^\star}$ and a node $w \in C$ we have both $C \in L$ and $C \setminus w \in L$, then*

$$\Theta(C, w) \in \operatorname{lin}(\Delta_L)$$

holds.

PROOF. The equation

$$\Theta(C, w) = \left(x^C, y^C\right) - \left(x^{C\setminus w}, y^{C\setminus w}\right)$$

shows this. □

Lemma 7 *Let L be a set of $n^\star$- and $(n^\star-1)$-cliques of $\mathcal{G}_{n^\star}$ and let w_1', w_2', w_3', $w_4' \in \mathcal{V}_{n^\star}$ be any nodes such that $\Upsilon(w_1', w_2', w_3', w_4')$ is defined. If there exists an $(n^\star-2)$-clique C in $\mathcal{G}_{n^\star}/\{w_1', w_2', w_3', w_4'\}$ such that $C \cup \{w_1', w_3'\} \in L$, $C \cup \{w_2', w_4'\} \in L$, $C \cup \{w_1'\} \in L$, $C \cup \{w_2'\} \in L$, $C \cup \{w_3'\} \in L$, and $C \cup \{w_4'\} \in L$, then*

$$\Upsilon(w_1', w_2', w_3', w_4') \in \operatorname{lin}(\Delta_L)$$

holds.

PROOF. This is due to

$$\begin{aligned}\Upsilon(w_1', w_2', w_3', w_4') = {} & \left(x^{C\cup\{w_1',w_3'\}}, y^{C\cup\{w_1',w_3'\}}\right) - \left(x^{C\cup\{w_1'\}}, y^{C\cup\{w_1'\}}\right) \\ & - \left(x^{C\cup\{w_3'\}}, y^{C\cup\{w_3'\}}\right) - \left(x^{C\cup\{w_2',w_4'\}}, y^{C\cup\{w_2',w_4'\}}\right) \\ & + \left(x^{C\cup\{w_2'\}}, y^{C\cup\{w_2'\}}\right) + \left(x^{C\cup\{w_4'\}}, y^{C\cup\{w_4'\}}\right).\end{aligned}$$

□

Lemma 8 *Let L be a set of $n^\star$- and $(n^\star-1)$-cliques of $\mathcal{G}_{n^\star}$, and let w_1, w_2, w_3, $w_4 \in \mathcal{V}_{n^\star}$ be any nodes such that $\Phi(w_1, w_2, w_3, w_4)$ is defined. If there exists an $(n^\star-3)$-clique C in $\mathcal{G}_{n^\star}/\{w_1, w_2, w_3, w_4\}$ such that $C \cup \{w_1, w_2\} \in L$, $C \cup \{w_2, w_3\} \in L$, $C \cup \{w_3, w_4\} \in L$, and $C \cup \{w_4, w_1\} \in L$, then*

$$\Phi(w_1, w_2, w_3, w_4) \in \operatorname{lin}(\Delta_L)$$

holds.

PROOF. This is obtained from

$$\Phi(w_1,w_2,w_3,w_4) = \left(x^{C\cup\{w_1,w_2\}}, y^{C\cup\{w_1,w_2\}}\right) - \left(x^{C\cup\{w_2,w_3\}}, y^{C\cup\{w_2,w_3\}}\right) + \left(x^{C\cup\{w_3,w_4\}}, y^{C\cup\{w_3,w_4\}}\right) - \left(x^{C\cup\{w_4,w_1\}}, y^{C\cup\{w_4,w_1\}}\right).$$

□

4 Affine Hulls, Dimensions, and Trivial Inequalities

After the preparations in Section 3, we now can treat the basic polyhedral questions. We perform the investigations for $\mathcal{QAP}^\star_{n^\star}$ first, and carry over the results to $\mathcal{QAP}_n$ afterwards.

4.1 Basic Facial Structure of $\mathcal{QAP}^\star_{n^\star}$

In Proposition 3.2 we have already analyzed the equation system (5), (6) holding for $\mathcal{QAP}^\star_{n^\star}$. It turned out that precisely one (arbitrary) equation is redundant in that system. The next theorem shows in particular that we do not have to search for more valid equations for $\mathcal{QAP}^\star_{n^\star}$.

Theorem 9 *Let $n^\star \geq 2$.*

(i) The affine hull of $\mathcal{QAP}^\star_{n^\star}$ is

$$\operatorname{aff}(\mathcal{QAP}^\star_{n^\star}) = \left\{ (x,y) \in \mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}} \,\middle|\, (x,y) \textit{ satisfies } (5), (6) \right\}.$$

(ii) The dimension of $\mathcal{QAP}^\star_{n^\star}$ is

$$\dim(\mathcal{QAP}^\star_{n^\star}) = \dim\left(\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}\right) - (n^\star(n^\star-1)-1).$$

(iii) The inequalities

$$y_e \geq 0 \qquad\qquad (e \in \mathcal{E}_{n^\star})$$

define facets of $\mathcal{QAP}^\star_{n^\star}$.

PROOF. By Proposition 4, part (ii) is implied by part (i). We will proceed as explained in Section 3.3 and prove (i) and (iii) together. Due to the symmetries of $\mathcal{QAP}^\star_{n^\star}$, it suffices to prove (iii) for $e = [n^\star, n^\star-1, n^\star-1, n^\star]$.

Let L be the set of all $n^\star$- and $(n^\star-1)$-cliques of $\mathcal{G}_{n^\star}$ that do not contain both nodes $(n^\star, n^\star-1)$ and $(n^\star-1, n^\star)$, i.e., L is the set of cliques belonging to the vertices of the face defined by $y_{[n^\star,n^\star-1,n^\star-1,n^\star]} \geq 0$. As in Section 3.3, we denote by Δ_L the set of differences of the incidence vectors belonging to L. We choose B to consist of E_{bas} (see Corollary 5), and take as the extra element the edge $e_0 = [n^\star, n^\star - 1, n^\star - 1, n^\star]$. Then we have to combine all nodes and edges starting from the vectors in $\mathcal{B}_0$ (the unit vectors belonging to $B \cup e_0$) and Δ_L in order to prove the theorem (since $y_{[n^\star,n^\star-1,n^\star-1,n^\star]} \geq 0$ defines a proper face).

We exhibit in three lemmas some of the vectors presented in Section 3.4 that are available for our proof. From now on, we will assume $n^\star \geq 5$. This simplifies the proof and does not really leave open a gap, because one can easily check the cases $n^\star \in \{2, 3, 4\}$ by computer, for example.

Lemma 10 *Let $w'_1, w'_2, w'_3, w'_4 \in \mathcal{V}_{n^\star}$ such that $\Upsilon(w'_1, w'_2, w'_3, w'_4)$ is defined. If neither $\{w'_1, w'_3\}$ nor $\{w'_2, w'_4\}$ is the edge $[n^\star, n^\star - 1, n^\star - 1, n^\star]$, then we have*

$$\Upsilon(w'_1, w'_2, w'_3, w'_4) \in \operatorname{lin}(\Delta_L) .$$

PROOF. Since $\mathcal{G}_{n^\star}/\{w'_1, w'_2, w'_3, w'_4\}$ has at least three rows and at least three columns, we can find an $(n^\star - 2)$-clique C of $\mathcal{G}_{n^\star}/\{w'_1, w'_2, w'_3, w'_4\}$ such that the nodes $(n^\star, n^\star - 1)$ and $(n^\star - 1, n^\star)$ are both not contained in C. Hence, Lemma 7 can be applied, yielding the claim. □

Lemma 11 *Let $w_1, w_2, w_3, w_4 \in \mathcal{V}_{n^\star}$ such that $\Phi(w_1, w_2, w_3, w_4)$ is defined. If all four edges $\{w_1, w_2\}$, $\{w_2, w_3\}$, $\{w_3, w_4\}$ and $\{w_4, w_1\}$ are different from $[n^\star, n^\star - 1, n^\star - 1, n^\star]$, then we have*

$$\Phi(w_1, w_2, w_3, w_4) \in \operatorname{lin}(\Delta_L) .$$

PROOF. There is at most one of the nodes $(n^\star, n^\star - 1)$ and $(n^\star - 1, n^\star)$ contained in $\{w_1, w_2, w_3, w_4\}$, hence we can assume (by a symmetry argument) that $(n^\star, n^\star - 1) \notin \{w_1, w_2, w_3, w_4\}$ holds. Since $\mathcal{G}_{n^\star}/\{w_1, w_2, w_3, w_4\}$ has at least two rows and at least two columns, we can find an $(n^\star - 3)$-clique C of $\mathcal{G}_{n^\star}/\{w_1, w_2, w_3, w_4\}$ with $(n^\star, n^\star - 1) \notin C$. Thus Lemma 8 yields the claim. □

Lemma 12 *Let $w \in \mathcal{V}_{n^\star}$ be any node. Then there is an $n^\star$-clique C of $\mathcal{G}_{n^\star}$, containing w, such that we have*

$$\Theta(C, w) \in \operatorname{lin}(\Delta_L) .$$

PROOF. This is due to Lemma 6, since $\mathcal{G}_{n^\star}/w$ has at least four rows and at least four columns, and hence, it is easy to find an $(n^\star - 1)$-clique of $\mathcal{G}_{n^\star}/w$ that does not contain a forbidden node. □

Now we combine all nodes and edges using Lemmas 10, 11, and 12. Let us partition the node set $\mathcal{V}_{n^\star}$ into four parts as indicated in the following table

$\times$	$\{1,2\}$	$\{3,\ldots,n^\star\}$
$\{1,2\}$	V_1	V_2
$\{3,\ldots,n^\star\}$	V_3	V_4

meaning that we have, e.g., $V_1 = \{1,2\}\times\{1,2\}$. Due to our assumption $n^\star \geq 5$, none of these four sets is empty, and $[n^\star, n^\star-1, n^\star-1, n^\star] \in V_4$. Recall that the *mate* of an edge $[i,j,k,l] \in \mathcal{E}_{n^\star}$ is the edge $[i,l,k,j]$. For any number $a \in \{1,2\}$ we denote by $\neg a$ the number with $\{a, \neg a\} = \{1,2\}$. We perform the necessary combinations in eight steps.

$\mathcal{E}_{n^\star}(V_1 \cup V_2)$: For every edge in $\mathcal{E}_{n^\star}(V_1 \cup V_2)$ either itself or its mate is contained in B. Hence, these edges can be combined by Lemma 10.

$\mathcal{E}_{n^\star}(V_1 \cup V_3)$: This is done analogously to the first step.

$(V_2 : V_3)$: Let $(i,j) \in V_2$ and $(k,l) \in V_3$, hence we have $i,l \in \{1,2\}$ and $j,k \in \{3,\ldots,n^\star\}$. Choosing $w_1 = (\neg i, l)$, $w_2 = (i, \neg l)$, $w_3 = (k,l)$, and $w_4 = (i,j)$ (see Figure 7), we can apply Lemma 11, yielding the desired combination of $[i,j,k,l] = \{w_3, w_4\}$, since the edges $\{w_1, w_2\}$, $\{w_2, w_3\}$ and $\{w_4, w_1\}$ are already combined.

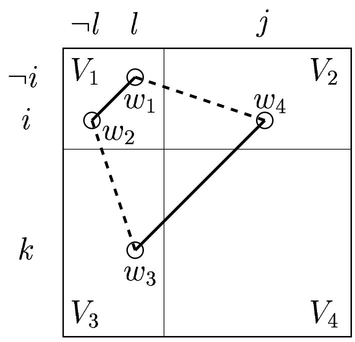


Fig. 7. Combination of the edges in $(V_2 : V_3)$.

$(V_1 : V_4)$: Since all edges in $(V_2 : V_3)$ are already combined, these edges can be combined by Lemma 10.

$(V_2 : V_4)$: Let $(i,j) \in V_2$ and $(k,l) \in V_4$, i.e., we have $i \in \{1,2\}$ and $j,k,l \in \{3,\ldots,n^\star\}$. We choose $w_1 = (\neg i, l)$, $w_2 = (i,1)$, $w_3 = (k,l)$, and $w_4 = (i,j)$ (see Figure 8), hence Lemma 11 applies and yields a combination of $[i,j,k,l] = \{w_3, w_4\}$, because, again, the edges $\{w_1, w_2\}$, $\{w_2, w_3\}$ and $\{w_4, w_1\}$ are already combined.

$(V_3 : V_4)$: These edges are combined analogously to the edges in $(V_1 : V_4)$.

$\mathcal{E}_{n^\star}(V_4)$: The edge $[n^\star, n^\star-1, n^\star-1, n^\star]$ is already combined since it the extra element. Let $[i,j,k,l] \in \mathcal{E}_{n^\star}(V_4) \setminus \{[n^\star, n^\star-1, n^\star-1, n^\star]\}$. The nodes $w_1 = (1,j)$, $w_2 = (k,1)$, $w_3 = (i,j)$, and $w_4 = (k,l)$ (see Figure 9) satisfy the conditions of Lemma 11, and thus, we can combine the edge

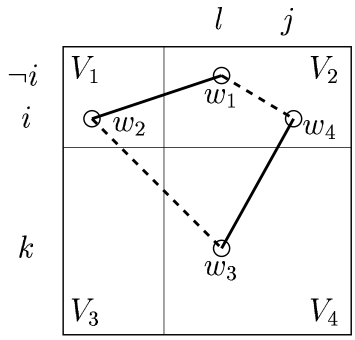


Fig. 8. Combination of the edges in $(V_2 : V_4)$.

$[i, j, k, l] = \{w_3, w_4\}$, because $\{w_1, w_2\}$, $\{w_2, w_3\}$ and $\{w_4, w_1\}$ have been combined in previous steps.

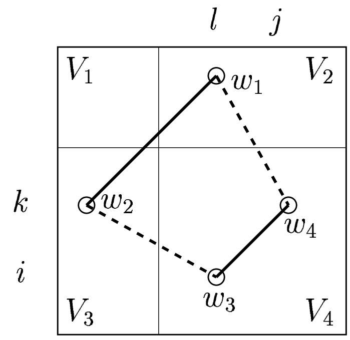


Fig. 9. Combination of the edges in $\mathcal{E}_n(V_4)$.

$\mathcal{V}_{\mathbf{n}^\star}$**:** Now that all edges are combined, it is easy to combine also the nodes using Lemma 12.

□

4.2 Basic Facial Structure of $\mathcal{Q}AP_n$

The next theorem shows that we also do not have to search for other equations for $\mathcal{Q}AP_n$ than those given by (1), ... ,(4). Furthermore, it describes possibilities to extract from that equation system a complete and non-redundant equation system for $\mathcal{Q}AP_n$. Parts (i) and (ii) of the theorem (as well as the results of Theorem 14) have independently been proved also by Padberg and Rijal [15,13]. However, the techniques we have developed in the previous sections allow us to give significantly simpler proofs here.

Theorem 13 *Let* $n \geq 3$.

(i) The affine hull of $\mathcal{Q}AP_n$ *is described by the equations* (1), ... ,(4), *i.e., a point* $(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ *is contained in* $\text{aff}(\mathcal{Q}AP_n)$ *if and only if it*

satisfies

$$
\begin{aligned}
x(\text{row}_i) &= 1 && (i \in \mathcal{N}) && (7)\\
x(\text{col}_j) &= 1 && (j \in \mathcal{N}) && (8)\\
-x_{(i,j)} + y((i,j) : \text{row}_k) &= 0 && (i,j,k \in \mathcal{N}, i \neq k) && (9)\\
-x_{(i,j)} + y((i,j) : \text{col}_l) &= 0 && (i,j,j \in \mathcal{N}, j \neq l). && (10)
\end{aligned}
$$

(ii) The dimension of the quadratic assignment polytope is

$$\dim(\mathcal{QAP}_n) = \dim\left(\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}\right) - (2n^3 - 5n^2 + 5n - 2).$$

(iii) Let $r, c \in \mathcal{N}$ be two row and column indices, respectively, and let $\mathcal{R}$ be a subset of the equations (7), ... ,(10) consisting precisely of

a) one equation from (7) or (8),

b) for all $(i,j) \in \mathcal{N} \setminus r \times \mathcal{N} \setminus c$ either (9) with one arbitrary $k \neq i$ or (10) with one arbitrary $l \neq j$,

c) all equations (9), (10) with $(i,j) = (r,c)$,

d) for all $i \in \mathcal{N} \setminus r$ the equation (9) with $(k,j) = (r,c)$,

e) for all $j \in \mathcal{N} \setminus c$ the equation (10) with $(i,l) = (r,c)$,

f) for all $(k,l) \in \mathcal{N} \setminus r \times \mathcal{N} \setminus c$ either (9) with $(i,j) = (r,l)$ or (10) with $(i,j) = (k,c)$,

g) for all pairs $\{i',k'\} \in \binom{\mathcal{N}\setminus r}{2}$ either (9) with $(i,j,k) = (i',c,k')$ or (9) with $(i,j,k) = (k',c,i')$,

h) for all pairs $\{j',l'\} \in \binom{\mathcal{N}\setminus c}{2}$ either (10) with $(i,j,l) = (r,j',l')$ or (10) with $(i,j,l) = (r,l',j')$,

i) either for one pair in g) or for one pair in h) the equation not yet chosen in g) or h), respectively

(where "either or" is always meant exclusively). Then removing $\mathcal{R}$ from the set of equations (7), ... , (10) yields a complete and non-redundant equation system for $\mathcal{QAP}_n$.

PROOF. In order to prove part (i), it suffices to show that the zero-liftings of (5) and (6) from $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$ into $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ can be linearly combined from the equation system (7), ... ,(10). This is sufficient due to the fact that then the solution space $\mathcal{A}$ of (7), ... ,(10), containing $\mathcal{QAP}_n$, is mapped by the projection $\pi^{(n)}$ isomorphically (see Proposition 1) into the solution space of (5) (6), which is the affine hull of $\mathcal{QAP}^\star_{n-1}$ (by Theorem 9). Hence, by the isomorphism between $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_{n-1}$, we have

$$\dim\left(\mathcal{QAP}^\star_{n-1}\right) = \dim(\mathcal{QAP}_n) \leq \dim(\mathcal{A}) \leq \dim\left(\mathcal{QAP}^\star_{n-1}\right),$$

showing that in particular $\text{aff}(\mathcal{QAP}_n) = \mathcal{A}$ must hold.

By symmetry arguments, we only need to show that the equation

$$x\left(\mathrm{row}_1^{(n)} \setminus (1,n) \cup \mathrm{row}_2^{(n)} \setminus (2,n)\right) - y\left(\mathrm{row}_1^{(n)} \setminus (1,n) : \mathrm{row}_2^{(n)} \setminus (2,n)\right) = 1 \tag{11}$$

is implied by (7), ... ,(10). We can obtain this by adding up the two equations (7) for $i = 1, 2$ as well as the two equations (9) with $j = n$ and $(i,k) \in \{(1,2),(2,1)\}$, subtract all equations (9) with $j \in \{1,\dots,n-1\}$ and $(i,k) \in \{(1,2),(2,1)\}$, and finally divide the obtained equation by two. Figure 10 illustrates the summation by showing three of its partial sums.

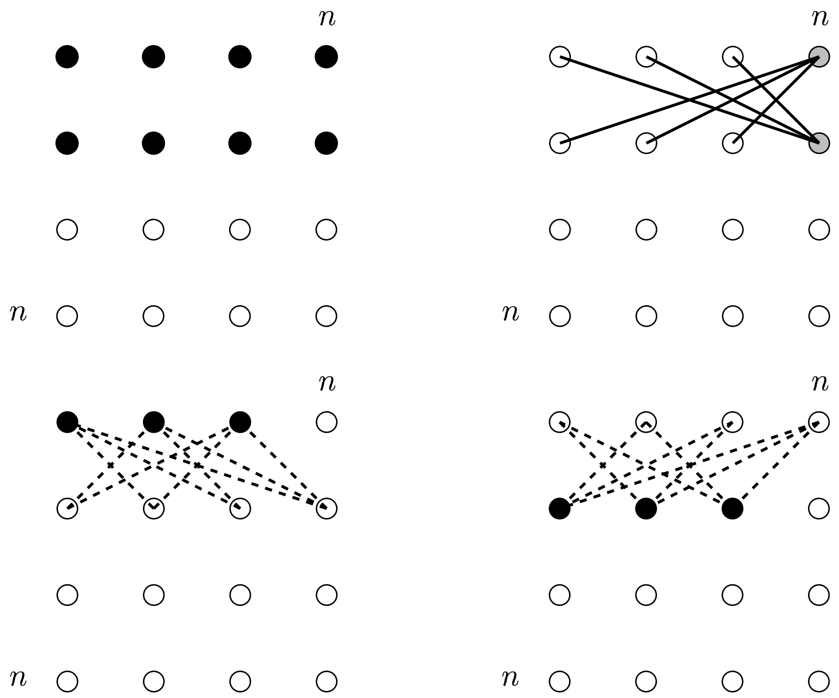


Fig. 10. Combination of equation (11).

Now, we will prove part (ii). When changing from $\mathcal{G}_{n-1}$ to $\mathcal{G}_n$ one obtains $2n-1$ new nodes, $2(n-1)^2(n-2)$ new edges connecting $\mathrm{row}_n^{(n)} \setminus (n,n)$ and $\mathrm{col}_n^{(n)} \setminus (n,n)$ with the old nodes, $(n-1)^2$ new edges between $\mathrm{row}_n^{(n)} \setminus (n,n)$ and $\mathrm{col}_n^{(n)} \setminus (n,n)$, and $(n-1)^2$ new edges from (n,n) to the old nodes, summing up to

$$\begin{aligned} 2n-1+2(n-1)^2(n-2)+2(n-1)^2 &= 2n-1+2(n-1)^3 \\ &= 2n-1+2n^3-6n^2+6n-2 \\ &= 2n^3-6n^2+8n-3 \end{aligned}$$

new items. Thus, we have (using Theorem 9)

$$\begin{aligned}
\dim(\mathcal{QAP}_n) &= \dim\left(\mathcal{QAP}^{\star}_{n-1}\right) \\
&= \dim\left(\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}\right) - ((n-1)(n-2)-1) \\
&= \dim\left(\mathbb{R}^{\mathcal{V}_{n}} \times \mathbb{R}^{\mathcal{E}_{n}}\right) - (2n^3-6n^2+8n-3) - (n^2-3n+1) \\
&= \dim\left(\mathbb{R}^{\mathcal{V}_{n}} \times \mathbb{R}^{\mathcal{E}_{n}}\right) - (2n^3-5n^2+5n-2),
\end{aligned}$$

proving part (ii).

It remains to prove part (iii). The cardinality of the set $\mathcal{R}$ of equations being removed from the system (7), ... ,(10) is

$$\begin{aligned}
|\mathcal{R}| &= 1 + (n-1)^2 + 2(n-1) + (n-1) + (n-1) \\
&\quad + (n-1)^2 + \frac{(n-1)(n-2)}{2} + \frac{(n-1)(n-2)}{2} + 1 \\
&= 2 + 2(n-1)^2 + 4(n-1) + (n-1)(n-2) \\
&= 2 + (n-1)(2(n-1) + 4 + (n-2)) \\
&= 2 + 3n(n-1) \\
&= 3n^2 - 3n + 2.
\end{aligned}$$

Hence, the remaining system consists of

$$\begin{aligned}
2n + 2n^2(n-1) - (3n^2-3n+2) &= 2n + 2n^3 - 2n^2 - 3n^2 + 3n - 2 \\
&= 2n^3 - 5n^2 + 5n - 2
\end{aligned}$$

equations. Due to part (ii) it suffices now to prove that this remaining system still has the same solution space as (7), ... ,(10). Hence, we will show how to obtain the equations in $\mathcal{R}$ as linear combinations of the ones in the remaining system.

In order to simplify the notations we will denote the equations (7) by x-row(i), (8) by x-col(j), (9) by xy-row(i,j,k), and (10) by xy-col(i,j,l). Due to symmetry reasons we can restrict to $r = n$ and $c = n$.

- **a)** We can obtain the equation removed from (7), (8) as a linear combination of the remaining ones, since this system has not full row rank, and hence, due to symmetry reasons, every single equation is redundant.
- **b)** For every fixed node $(i,j) \in \mathcal{V}_n$ adding up all equations xy-row(i,j,k) yields the same as adding up all equations xy-col(i,j,l). Hence, for every fixed node $(i,j)\mathcal{V}_n$ the system of equations

  $$\{\text{xy-row}(i,j,k) \mid k \in \mathcal{N} \setminus i\} \cup \{\text{xy-col}(i,j,l) \mid l \in \mathcal{N} \setminus j\}$$

  has not full row rank. Thus there must be at least one redundant equation among them. Due to symmetry reasons, again, this must be an arbitrary one. But for $(i,j) \in \mathcal{V}_n \setminus (\mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)})$ the set $\mathcal{R}$ contains only one of these equations that therefore can be obtained as a linear combination of the remaining ones.

f) Suppose an equation xy-row(n,j,k) with $k,j \in \mathcal{N} \setminus n$ is contained in $\mathcal{R}$. Then xy-col(k,n,j) is not contained in $\mathcal{R}$, and furthermore, we can use all xy-row(i,j,k) and xy-col(i,j,l) with $i,j \in \{1,\ldots,n-1\}$ for the linear combination since they have already been obtained as linear combinations in b). Adding up all xy-col(k,l,j) for $l \in \mathcal{N} \setminus j$, subtracting all xy-row$(i,j,k)$ for $i \in \mathcal{N} \setminus \{k,n\}$, and finally adding x-row(k) and subtracting x-col(j) yields a combination of xy-row(n,j,k) (see Figure 11). An equation xy-col(i,n,l) contained in $\mathcal{R}$ can be obtained as a linear

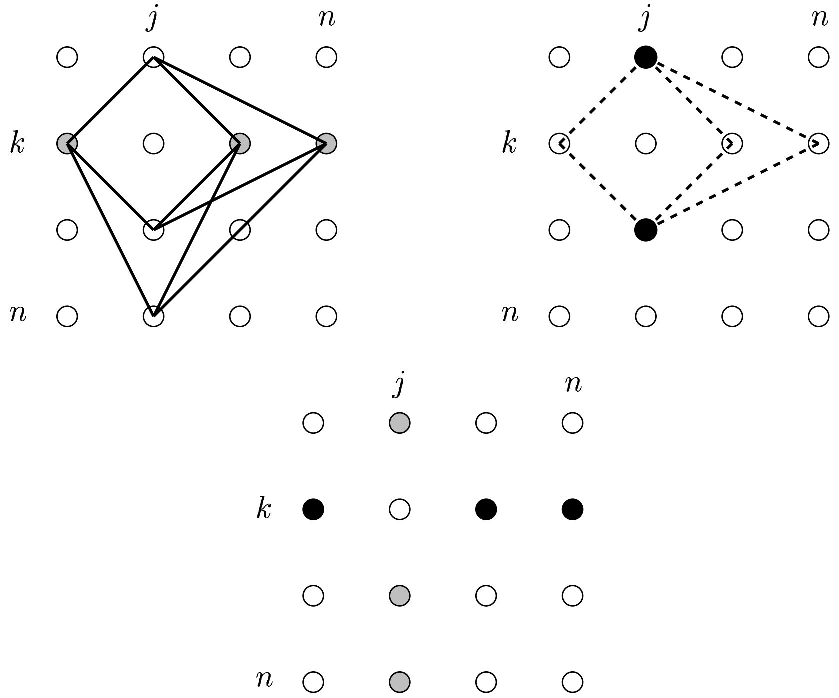


Fig. 11. Combination of the equations removed in f).

combination analogously.

g),h),i) If an equation xy-row(i,n,k) with $i,k \in \mathcal{N} \setminus n$ and $i \neq k$ is not contained in $\mathcal{R}$, then we can obtain the equation

$$x\left(\mathrm{row}_i^{(n)} \setminus (i,n) \cup \mathrm{row}_k^{(n)} \setminus (k,n)\right) - y\left(\mathrm{row}_i^{(n)} \setminus (i,n) : \mathrm{row}_k^{(n)} \setminus (k,n)\right) = 1 \quad (12)$$

as a linear combination by just using equations from the remaining system and equations that have already been obtained as linear combinations in a) and b) (see Figure 12). The same holds also for any equation xy-col(n,j,l) with $j,l \in \mathcal{N} \setminus n$ and $j \neq l$. We can restrict to the case, where the equation chosen in g) is xy-row$(1,n,2)$. Thus, this way we can

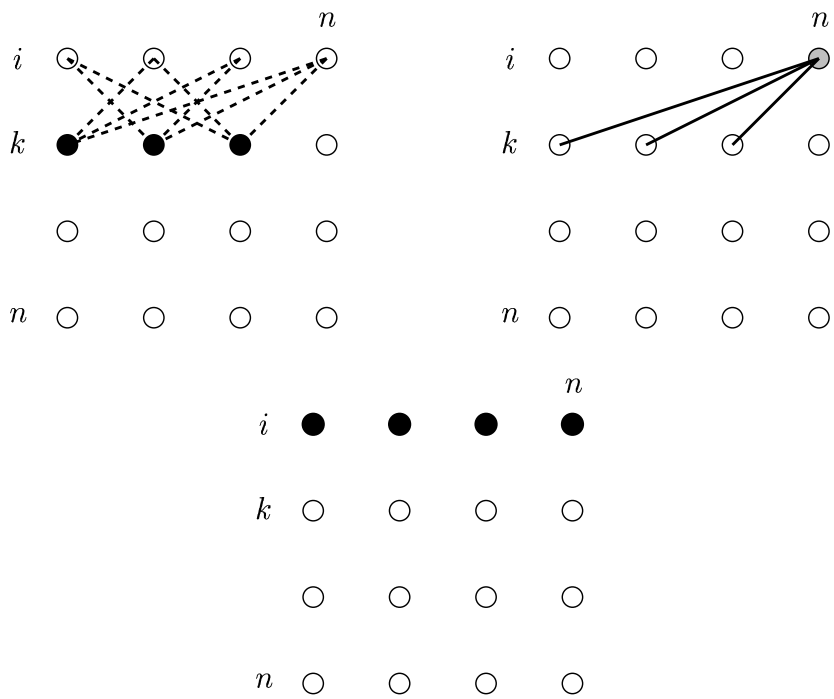


Fig. 12. Combination of equation (12).

obtain as linear combinations all equations that are the zero-lifting of (5) and (6) except

$$x\left(\text{row}_1^{(n)} \setminus (1,n) \cup \text{row}_2^{(n)} \setminus (2,n)\right) - y\left(\text{row}_1^{(n)} \setminus (1,n) : \text{row}_2^{(n)} \setminus (2,n))\right) = 1.$$

However, this one can be obtained as a linear combination of the other zero-lifted equations due to the fact that one arbitrary equation in (5), (6) is redundant (see Proposition 4). Proceeding "backwards" now yields also the equation xy-row$(1,n,2)$ that was put into $\mathcal{R}$ in g).

Hence, in the subsequent arguments, we can use for every pair $\{i,k\} \in \mathcal{N} \setminus n$ with $i \neq k$ one equation of xy-row(i,n,k) and xy-row(k,n,i) as well as for every pair $\{j,l\} \in \mathcal{N} \setminus n$ with $j \neq l$ one equation of xy-col(n,j,l) and xy-col(n,l,j).

An equation xy-row(i,n,k) with $i,k \in \mathcal{N} \setminus n$ and $i \neq k$ that is contained in $\mathcal{R}$ can now be obtained as a linear combination by adding up all xy-row(k,j,i) for all $j \in \mathcal{N}$, subtracting all xy-row(i,j,k) for $j \in \mathcal{N} \setminus n$, adding x-row$(k)$, and subtracting x-row$(i)$ (see Figure 13). Analogously, one treats an equation xy-col(n,j,l) with $j,l \in \mathcal{N} \setminus n$ that is contained in $\mathcal{R}$, where xy-col(n,l,j) is not contained in $\mathcal{R}$, and thus, all equations removed in g), h) and i) are obtained as linear combinations.

d) We obtain every equation xy-row(i,n,n) with $i \in \mathcal{N} \setminus n$ as a linear combination by adding up all xy-col(i,n,l) for all $l \in \mathcal{N} \setminus n$ and subtracting all xy-row(i,n,k) for all $k \in \mathcal{N} \setminus \{i,n\}$ (see Figure 14).

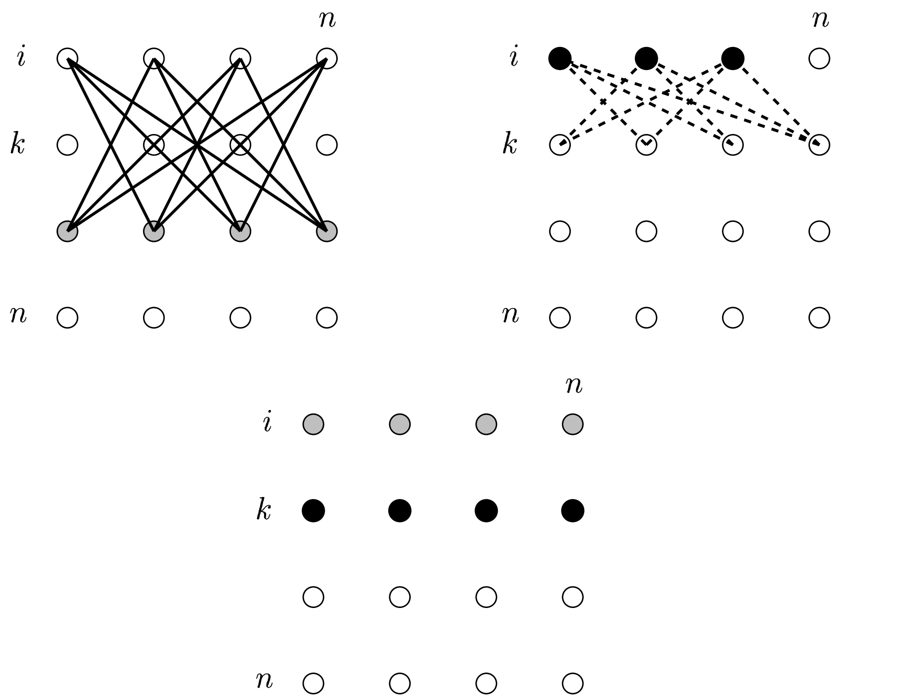


Fig. 13. Combination of the equations removed in g) and h).

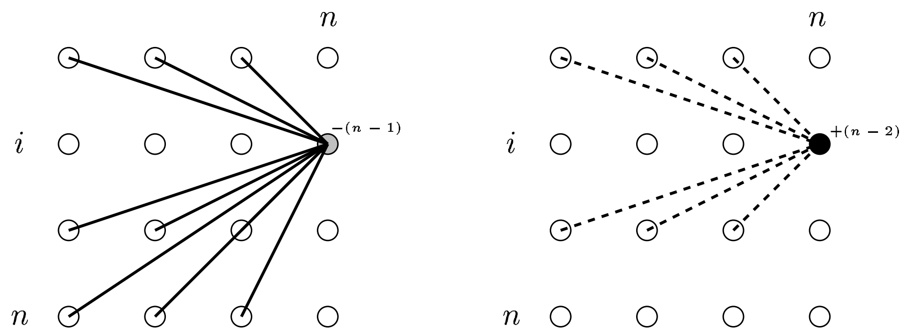


Fig. 14. Combination of the equations removed in d).

e) Here, we can proceed analogously to d).

c) We obtain an equation xy-row(n, n, k) by adding up all xy-row(k, j, n) for $j \in \mathcal{N}$, subtracting all xy-row(n, j, k) for $j \in \mathcal{N} \setminus n$, adding x-row$(k)$, and finally subtracting x-row(n) (this is the same procedure as for the combination of the equations removed in g) and h), see Figure 13). Finally, the equations xy-col(n, n, l) are obtained as linear combinations analogously.

□

We close this treatment of the basic questions concerning redundancies in the linear constraints we have considered so far by a classification of the trivial inequalities for $\mathcal{Q}AP_n$.

Theorem 14 *Let* $n \geq 3$.

(i) The inequalities

$$y_e \geq 0 \qquad (e \in \mathcal{E}_n)$$

define facets of $\mathcal{Q}AP_n$.

(ii) The inequalities

$$\begin{array}{ll} y_e \leq 1 & (e \in \mathcal{E}_n) \\ x_v \geq 0 & (v \in \mathcal{V}_n) \\ x_v \leq 1 & (v \in \mathcal{V}_n) \end{array}$$

are implied by the equations (7), ... ,(10) and the nonnegativity constraints $y \geq 0$ *on the edge variables.*

PROOF. Part (i) follows immediately from part (iii) of Theorem 9. In order to prove part (ii), observe that (9), e.g., yields from $y \geq 0$ also the nonnegativity of x. From this one obtains, e.g. by (7), that $x \leq 1$ holds, and this leads, exploiting once more (9) and $y \geq 0$, to $y \leq 1$. □

5 Conclusions

With the introduction of the "star-transformation" of the quadratic assignment polytope, now a technique is available that allows to perform deeper polyhedral investigations of the quadratic assignment problem. This certainly provides the possibility of investigating large classes of inequalities with respect to the question if they define facets of the quadratic assignment polytope. Considering, e.g., the fact that polyhedral investigations of the traveling salesman problem have lead to algorithms that now can solve instances with several thousands of cities, we hope that the techniques that we have presented in this paper will give a key to utilize the large potential that polyhedral investigations of the quadratic assignment problem have for improving the practical solution of this "extremely" hard combinatorial optimization problem.

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