Chapter 1

POLYHEDRAL METHODS FOR THE QAP

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Abstract For many combinatorial optimization problems investigations of associated polyhedra have led to enormous successes with respect to both theoretical insights into the structures of the problems as well as to their algorithmic solvability. Among these problems are quite prominent $\mathcal{NP}$-hard ones, like, e.g., the traveling salesman problem, the stable set problem, or the maximum cut problem. In this chapter we overview the polyhedral work that has been done on the quadratic assignment problem (QAP). Our treatment includes a brief introduction to the methods of polyhedral combinatorics in general, descriptions of the most important polyhedral results that are known about the QAP, explanations of the techniques that are used to prove such results, and a discussion of the practical results obtained by cutting plane algorithms that exploit the polyhedral knowledge. We close by some remarks on the perspectives of this kind of approach to the QAP.

1. INTRODUCTION

Polyhedral combinatorics is a branch of combinatorics and, in particular, of combinatorial optimization that has become quite broad and successful since it has sprouted in the 50's and 60's. Its scope is the treatment of combinatorial (optimization) problems by methods of (integer) linear programming. Besides many beautiful results on several polynomially solvable combinatorial optimization problems, methods of polyhedral combinatorics have also led to enormous progress in the practical solvability of many $\mathcal{NP}$-hard optimization problems. The most prominent example might be the traveling salesman problem, where today many instances with several thousand cities can be solved — by extremely elaborate exploitation of polyhedral results on the problem.

Among further examples one finds the maximum cut problem, the linear ordering problem, or the stable set problem. While these problems have been investigated by polyhedral methods extensively since the 70's, nearly no polyhedral results on the QAP were known until a few years ago. In this chapter, we give an overview on the (theoretical and practical) results on the QAP that have been obtained since then. In particular, we explain some crucial techniques that made possible to prove these results and that may be also useful for deriving further results in the future.

The objects of polyhedral combinatorics are combinatorial optimization problems with *linear* objective functions, while the QAP, e.g., in its original formulation by Koopmans and Beckmann, 1957 as the task to find a permutation π that minimizes $\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{\pi(i)\pi(k)} + \sum_{i=1}^{n} c_{i\pi(i)}$ (with $A = (a_{ik}) \in \mathbb{Q}^{n\times n}$ the *flow-*, $B = (b_{jl}) \in \mathbb{Q}^{n\times n}$ the *distance-*, and $C = (c_{ij}) \in \mathbb{Q}^{n\times n}$ the *linear cost-matrix*), has a quadratic objective function. However, there are lots of different equivalent formulations of the problem with linear objective functions. The one that turns out to yield a suitable starting point for a polyhedral treatment is due to Lawler, 1963. He formulated the QAP by representing the permutations π by *permutation matrices* $X = (x_{ij}) \in \{0,1\}^{n\times n}$ in the following way:

$$
\begin{array}{llrcll}
\text{(L)} & \min & \displaystyle\sum_{i,k=1}^{n}\sum_{j,l=1}^{n} q_{ijkl}y_{ijkl} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & & & \\
 & \text{s.t.} & \displaystyle\sum_{j=1}^{n} x_{ij} & = & 1 & (i \in \{1,\ldots,n\}) \\
 & & \displaystyle\sum_{i=1}^{n} x_{ij} & = & 1 & (j \in \{1,\ldots,n\}) \\
 & & y_{ijkl} & = & x_{ij}x_{kl} & (i,j,k,l \in \{1,\ldots,n\}) \\
 & & x_{ij} & \in & \{0,1\} & (i,j \in \{1,\ldots,n\})
\end{array}
$$

Of course, in case of a QAP instance of Koopmans & Beckmann type one takes $q_{ijkl} = a_{ik}b_{jl}$.

Several proposals have been made in the literature to replace the nonlinear constraints $y_{ijkl} = x_{ij}x_{kl}$ by some linear equations and inequalities and to derive lower bounding procedures from these different types of *linearizations*. Adams and Johnson, 1994 (see also Johnson, 1992) gave the following formulation (notice that every solution to (L) satisfies $y_{ijkj} = 0$

for $i \neq k$, $y_{ijil} = 0$ for $j \neq l$, and $y_{ijij} = x_{ij}$):

$$
\begin{array}{llrcll}
\text{(AJ)} & \min & \multicolumn{4}{l}{\displaystyle\sum_{\substack{i,k=1\\ i\neq k}}^{n} \sum_{\substack{j,l=1\\ j\neq l}}^{n} q_{ijkl} y_{ijkl} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}} \\
& \text{s.t.} & \displaystyle\sum_{j=1}^{n} x_{ij} & = & 1 & (i \in \{1,\dots,n\}) \\
& & \displaystyle\sum_{i=1}^{n} x_{ij} & = & 1 & (j \in \{1,\dots,n\}) \\
& & \displaystyle\sum_{\substack{l=1\\ l\neq j}}^{n} y_{ijkl} & = & x_{ij} & (i,j,k \in \{1,\dots,n\}, i \neq k) \\
& & \displaystyle\sum_{\substack{k=1\\ k\neq i}}^{n} y_{ijkl} & = & x_{ij} & (i,j,l \in \{1,\dots,n\}, j \neq l) \\
& & y_{ijkl} & = & y_{klij} & (i,j,k,l \in \{1,\dots,n\}, i < k, j \neq l) \\
& & y_{ijkl} & \geq & 0 & (i,j,k,l \in \{1,\dots,n\}, i \neq k, j \neq l) \\
& & x_{ij} & \in & \{0,1\} & (i,j \in \{1,\dots,n\})
\end{array}
$$

Additionally, they showed that this formulation yields a stronger *linear programming relaxation* (obtained from the formulation by relaxing the integrality constraints $x_{ij} \in \{0,1\}$ to $0 \leq x_{ij} \leq 1$) than most other linearizations available in the literature. In particular, the lower bounds obtained from solving the corresponding linear programs are at least as good as several well-known lower bounds, including the most popular one proposed by Gilmore, 1962 and Lawler, 1963.

Resende et al., 1995 have done extensive computational studies with the lower bounds obtained from the linear programming relaxation of (AJ). It turned out that this yields rather good bounds in practice. The polyhedral approach to the QAP can be viewed as the attempt to sharpen these bounds by insights derived from investigations of the geometric structure of the feasible solutions to (L).

In order to provide the reader who is not yet acquainted with the subject of polyhedral combinatorics with the necessary background, we give a short introduction into this field in Section 2.. In Section 3. we define the central objects used to describe QAPs in terms of polyhedral combinatorics: the different versions of *QAP-polytopes*. A very useful technique for working with these polytopes is explained in Section 4.. Section 5. contains the most important results that are known on the QAP-polytopes, and Section 6. reports on practical results obtained by cutting plane algorithms that exploit these results. We close the chapter by a discussion of the future perspectives of the polyhedral approach to the QAP in Section 7..

2. POLYHEDRAL COMBINATORICS

This short section can only touch the basic principles of polyhedral combinatorics with respect to those parts of the theory that we need for the later work on the QAP. There are many concepts and results besides the scope of this work. For an overview we refer to Schrijver, 1995, for comprehensive treatments to Schrijver, 1986 and Nemhauser and Wolsey, 1988, and for a textbook Wolsey, 1998.

Combinatorial Optimization and Linear Programming. The subjects of polyhedral combinatorics are, in general, (linear) combinatorial optimization problems of the following kind. Let U be a finite set, $c \in \mathbb{R}^U$ an *objective function vector*, and let a subset $\mathcal{F} \subseteq 2^U$ of the subsets of U be specified, the *feasible solutions*. The *objective function value* of a feasible solution $F \in \mathcal{F}$ is $c(F) = \sum_{u \in F} c_u$. The problem we are interested in is to find a feasible solution $F_{\text{opt}} \in \mathcal{F}$ with

$$c(F_{\text{opt}}) = \min\{c(F) : F \in \mathcal{F}\} \ . \tag{1.1}$$

In the light of the formulation (L) (see Section 1.), it is clear that the QAP falls into this class.

Using the concept of the *incidence vector* χ^F of a feasible solution $F \in \mathcal{F}$ defined via

$$\chi_u^F = \begin{cases} 1 & \text{if } u \in F \\ 0 & \text{otherwise} \end{cases}$$

(i.e., χ^F is the characteristic vector of $F \subseteq U$), solving (1.1) is equivalent to finding $F_{\text{opt}} \in \mathcal{F}$ with

$$c^T \chi^{F_{\text{opt}}} = \min\{c^T \chi^F : F \in \mathcal{F}\} \ ,$$

i.e., equivalent to the task of minimizing a linear function over a finite set of vectors. This is equivalent (forgetting for a moment the minimizing element that we also want to find) to minimizing that linear function over the convex hull

$$\begin{aligned} \mathcal{P}_{\mathcal{F}} &= \operatorname{conv}\{\chi^F : F \in \mathcal{F}\} \\ &= \left\{ \sum_{F \in \mathcal{F}} \lambda_F \chi^F : \sum_{F \in \mathcal{F}} \lambda_F = 1, \lambda_F \geq 0 \text{ for all } F \in \mathcal{F} \right\} \end{aligned}$$

of these vectors, called the *associated polytope* to that combinatorial optimization problem.

The background for polyhedral combinatorics is a theorem found by Minkowski, 1896 and Weyl, 1935, who proved that a subset of a vector space $\mathbb{R}^n$ is the convex hull of a finite set of vectors if and only if it is the bounded solution space of a finite system of linear equations and inequalities. Hence, (writing every equation as two inequalities) a finite system $Ax \leq b$ of linear inequalities exists such that

$$\mathcal{P}_{\mathcal{F}} = \{x \in \mathbb{R}^U : Ax \leq b\}$$

holds.

Polyhedral combinatorics can be described as the discipline that tries to *find* such systems (called *linear descriptions*) for special polytopes arising from combinatorial optimization problems in the way described above, and to exploit these systems in order to derive structural insights into the problems they describe as well as algorithms to solve them. Once a linear system for a given problem is found, the theory of *linear programming* (for introductions see, e.g., Chvátal, 1983; Padberg, 1995) provides the tools for its exploitation in the above sense.

Linear programming deals with optimizing linear functions over (finite) systems of linear equations and inequalities. One of its central results is the *strong duality theorem*, stating that the *linear programs*

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \leq & b \end{array}$$

and

$$\begin{array}{lrcl} \max & b^T y & & \\ \text{s.t.} & A^T y & = & c \\ & y & \geq & 0 \end{array}$$

(called *dual* to each other) have the same optimal solution value (if both optimal values exist). If one succeeds in describing the polytope associated with a combinatorial optimization problem by a system of linear equations and inequalities, then the strong duality theorem of linear programming usually leads to a "short certificate" in the sense of Edmonds, 1965a and Edmonds, 1965b, i.e., the guarantee that for an optimal solution to the problem there is a polynomially sized proof of its optimality. In other words, the corresponding decision problem (supposed to be contained in $\mathcal{NP}$) is contained in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$.

In fact, in many cases, finding such a linear description even yields that the combinatorial optimization problem under investigation is solvable in polynomial time. This is due to the work of Grötschel et al., 1981, Karp and Papadimitriou, 1982 and Padberg and Rao, 1980. After Khachiyan, 1979 had shown that the ellipsoid method can be used to solve linear programs in polynomial time, they extended this result by proving that a linear program

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \leq & b \end{array}$$

can basically be solved in time depending polynomially on the running time of a *separation procedure.* A separation procedure for a (maybe only implicitly given) linear system $Ax \leq b$ finds for every point $x^\star$ an inequality in that system that is violated by $x^\star$ if one exists and otherwise asserts that $x^\star$ satisfies the whole system, i.e., it solves the *separation problem* for the system $Ax \leq b$.

Thus, finding an arbitrarily large complete linear description for a polytope associated with some combinatorial optimization problem and proving that the separation problem for this system is polynomially solvable (in the input size of the original problem) immediately yields that the optimization problem is polynomially solvable, too. In fact, also the opposite direction is true (see Grötschel et al., 1988). For every polynomially solvable combinatorial optimization problem the separation problem belonging to a complete linear description of the associated polytope is solvable in polynomial time. These results are usually known as the *(polynomial) equivalence of optimization and separation.*

Proving that the underlying combinatorial optimization problem is polynomially solvable via showing that the associated linear programs are solvable in polynomial time is only one possibility that opens up if one succeeds in describing the respective polytope completely by a linear system. Another one might be derived directly from the strong duality theorem, if the dual linear program can also be interpreted as some combinatorial optimization problem. Such a combinatorial *min-max relation* often leads to combinatorial algorithms for the investigated problems.

Hence, for several reasons, there is a strong interest in finding linear systems that describe polytopes coming from combinatorial optimization problems. Several beautiful techniques have been developed for this purpose, e.g., the concepts of *total unimodularity*, *total dual integrality*, or the theory of *blocking-* and *anti-blocking polyhedra.*

However, we do not concentrate here on these parts of polyhedral combinatorics, since pursuing the goal of finding "useful" complete linear descriptions of the polytopes associated with the QAP (or any other $\mathcal{NP}$-hard combinatorial optimization problem) is not very promising. This is due to the above remark that finding such a description usually yields a "good characterization" of the underlying problem, i.e., it would lead to $\mathcal{NP} = \text{co-}\mathcal{NP}$ in our case, what is not to expect. In fact, Karp and Papadimitriou, 1982 proved that, unless $\mathcal{NP} = \text{co-}\mathcal{NP}$ holds, no linear description of a polytope associated in the above way with an $\mathcal{NP}$-hard combinatorial optimization problem can exist with the property that for every inequality in the description its validity for the whole polytope can be proved in time polynomially depending on the size of the problem. Consequently, a "useful" description of the linear system describing a polytope coming from such a problem is unlikely to exist.

It follows from the previous discussion that we can only expect o find *partial* linear descriptions of the polytopes associated with the QAP. Notice that the "impossibility" of finding complete descriptions of polytopes coming from $\mathcal{NP}$-hard problems is only due to the inequalities. From the complexity point of view, there are no reasons against finding a *complete equation system* for such a polytope, i.e., a set of equations whose solution space is precisely the affine hull of the polytope.

What is a partial description good for? The aim of finding (as tight as possible) partial descriptions of polytopes coming from $\mathcal{NP}$-hard optimization problems is the computation of lower bounds (in case of a minimization problem) on the (yet unknown) optimal solution value of an instance. Lower bounds are very important tools in combinatorial optimization at all (regardless of polyhedral combinatorics). One reason for their importance is that they allow to give quality guarantees for feasible solutions that might have been obtained by heuristics, i.e., algorithms that do not necessarily give *optimal* solutions. Furthermore, they can be incorporated into implicitly enumerative algorithms like, e.g., branch-and-bound methods. There they can, applied to a problem defined on a subset of the solutions $\mathcal{F}$, give the guarantee that the overall optimal solution is not contained in that subset, and hence, one does not have to search this subset.

By solving the linear programs arising from a partial description, one clearly obtains lower bounds for the respective problem. However, usually also the partial descriptions contain exponentially many inequalities that cannot be handled by any linear programming solver in praxis. The way out is to develop *separation procedures* similar to the subroutines mentioned in the context of the ellipsoid method, which try to find for a given point $x^\star$ inequalities in the partial description that are violated by $x^\star$, i.e., which *cut off* $x^\star$. Such inequalities (or more precisely, the boundary hyperplanes of the halfspaces defined by them) are called *cutting planes.*

Now, one starts by solving a small linear program containing only some inequalities from the partial linear description. Often one takes an *integer programming formulation* here, i.e., a set of inequalities whose integer solutions are precisely the incidence vectors of the problem. After the linear program is solved, one checks, if, by chance, the resulting optimal solution vector $x^\star$ is an incidence vector of a feasible solution $F \in \mathcal{F}$. If this happens to be true, then, clearly, F must be an optimal solution. If the solution vector is not an incidence vector, then one calls the separation procedure in order to find inequalities that are violated by $x^\star$. If the procedure is successful, then the detected inequalities are added to the linear program, which is resolved, potentially giving a better bound. Iterating this process, one obtains a *cutting plane algorithm* for computing lower bounds for the problem. A branch-and-bound algorithm using this kind of lower bounding procedure is called a *branch-and-cut algorithm.*

If the pure cutting plane bounding procedure ends already with a bound that guarantees a known feasible solution to be optimal, one has obtained a very nice result, namely, a short, i.e., polynomially sized, *certificate* for the optimality of that solution. This is due to the fact that the final linear program of the cutting plane run provides a proof for optimality, in this case, and linear programming theory yields that one can remove equations and inequalities from every linear program such that the remaining linear program still has the same optimal solution value and its number of constraints does not exceed its number of variables. Hence, although the cutting plane algorithm may have run for a very long time, it might yield (if one is lucky) after all a short certificate either for the optimality or at least for a certain quality of the known solution (see, e.g., Jünger et al., 1994 for provably good solutions for the example of the *traveling salesman problem*).

Due to efficiency reasons, one desires in particular to find non-redundant (partial) linear descriptions. For the equations in such a partial description this means that one wants to avoid that any among them is a linear combination of some others. For the inequalities analogous redundancies coming from linear combinations with nonnegative coefficients for inequalities (and arbitrary ones for equations) should be excluded. The questions concerning redundancies in the (partial) linear descriptions are strongly related to the geometry of the polytope associated with the combinatorial optimization problem. They lead to asking for the *dimension* of the polytope or for the possibility of proving that a certain inequality is unavoidable in a complete description of the polytope (and hence also cannot be redundant in any partial linear description). This is the point, where polyhedral theory takes over.

Polyhedral Theory. Used in an auxiliary way, only a few aspects of the theory of polytopes become apparent in the context of our polyhedral investigations on the QAP. However, the general theory of polytopes is a fascinating and strongly developing field. For entering this wonderful area of mathematics as well as for the proofs of the statements in this section, we recommend the classical book of Grünbaum, 1967, as well as Ziegler, 1995 and Klee and Kleinschmidt, 1995.

Polytopes can (that was the central point in the previous section) equivalently be defined as the convex hulls of finite point sets in $\mathbb{R}^n$ or as the bounded solution spaces of finite systems of linear inequalities (and equations), i.e., as the bounded intersections of finitely many halfspaces (and hyperplanes) of $\mathbb{R}^n$. Two concrete examples are the *n-hypercube*

$$\begin{aligned} C_n &= \operatorname{conv}\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_i\in\{0,1\}\} \\ &= \big\{(x_1,\dots,x_n)\in\mathbb{R}^n : 0\le x_i\le 1 \ (1\le i\le n)\big\} \end{aligned}$$

and the *standard-*$(n-1)$*-simplex*

$$\begin{aligned} \Delta_{n-1} &= \operatorname{conv}\{e_1,\dots,e_n\} \\ &= \left\{(x_1,\dots,x_n)\in\mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = 1, x_i\ge 0 \ (1\le i\le n)\right\} \end{aligned}$$

(see Figure 1.1).

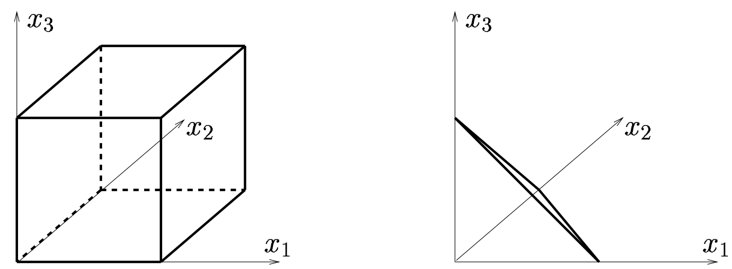

Figure 1.1 The 3-hypercube and the standard-2-simplex.

Two polytopes $\mathcal{P}\subseteq\mathbb{R}^n$ and $\mathcal{Q}\subseteq\mathbb{R}^m$ are *affinely isomorphic* if there is an affine map $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ (not necessarily an affine transformation) that induces a bijection between the points of $\mathcal{P}$ and $\mathcal{Q}$, or, equivalently, between the vertices of $\mathcal{P}$ and $\mathcal{Q}$. Whenever we say that two polytopes are *isomorphic* it is meant that they are affinely isomorphic.

If $\mathcal{P}\subseteq\mathbb{R}^n$ is a polytope, then its *dimension* $\dim(\mathcal{P})$ is defined as the dimension of its affine hull (i.e., the linear dimension of the linear subspace belonging to $\operatorname{aff}(\mathcal{P})$). If $\dim(\mathcal{P})=n$ holds, then $\mathcal{P}$ is called *full-dimensional*. We call the difference between the dimension of the vector space where the polytope is defined in and the dimension of the polytope the *dimensional gap*. The dimensional gap equals the minimal number of equations in a *complete equation system* for $\mathcal{P}$, i.e., an equation system having the affine hull of $\mathcal{P}$ as its solution space.

If a halfspace

$$\mathcal{H} = \{x \in \mathbb{R}^n : a^T x \leq \alpha\}$$

of $\mathbb{R}^n$ with the *boundary hyperplane*

$$\mathcal{B} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$$

contains the polytope $\mathcal{P}$, then $\mathcal{P} \cap \mathcal{B}$ is called a *face* of the polytope $\mathcal{P}$. Both the halfspace $\mathcal{H}$ as well as the inequality $a^T x \leq \alpha$ are said to *define* that face of $\mathcal{P}$. Adding any equation that is valid for the whole polytope to an inequality that defines a face of it yields another inequality defining the same face.

By the characterization of polytopes as the bounded intersections of finitely many halfspaces and hyperplanes, every face of a polytope is a polytope, again. Notice that both the polytope $\mathcal{P}$ itself as well as the empty set are faces of $\mathcal{P}$. The other faces of $\mathcal{P}$ are called *proper faces.* If $\mathcal{F}_1$ is a face of the polytope $\mathcal{P}$, and $\mathcal{F}_2$ is a face of $\mathcal{F}_1$, then $\mathcal{F}_2$ is also a face of $\mathcal{P}$.

Faces of $\mathcal{P}$ that have dimension 0, 1, $\dim(\mathcal{P}) - 2$, or $\dim(\mathcal{P}) - 1$ are called *vertices*, *edges*, *ridges*, and *facets* of $\mathcal{P}$, respectively. Facets of full-dimensional polytopes have the convenient property to be described by unique (up to multiplications with positive scalars) inequalities. Like every edge is the convex hull of two uniquely determined vertices, every ridge is the intersection of two uniquely determined facets. The following results are very basic for polyhedral theory.

(i) Every polytope is the convex hull of its vertices. If a polytope is the convex hull of some finitely many points, then its vertices must be among them (in particular, a polytope has finitely many vertices).

(ii) The vertices of a face $\mathcal{F}$ of a polytope $\mathcal{P}$ are precisely the vertices of $\mathcal{P}$ that are contained in $\mathcal{F}$ (in particular, every polytope has only finitely many faces).

(iii) Every polytope is the intersection of all halfspaces that define facets of it and all hyperplanes it is contained in. If a polytope is the intersection of some halfspaces (and hyperplanes), then all halfspaces that define facets of it must be among them.

(iv) Every face of a polytope $\mathcal{P}$ is the intersection of some facets of $\mathcal{P}$.

Point (iii) shows in particular that in any intersection of pairwise distinct facet defining halfspaces of a polytope there is no redundant halfspace. In other words, every (partial) linear description of a polytope $\mathcal{P}$ that contains (besides the equations) only inequalities that define (pairwise distinct) facets of $\mathcal{P}$ is non-redundant, as long as the equations in it are non-redundant. Hence, aiming to find non-redundant (partial) descriptions of polytopes, one should concentrate on facet defining inequalities. Clearly, keeping the equation system in the (partial) linear description non-redundant means that the corresponding matrix should have full row rank.

There is another important property of facets from the point of view of polyhedral combinatorics. Once one has found a class of inequalities contributing to a partial description of the polytope under investigation, one might (even knowing already that it is non-redundant with respect to that partial description) ask if it is possible to improve that new class by, e.g., "playing with the coefficients". Only the fact that the new inequalities define facets of the polytope can tell us at which point we do not have to try to strengthen our inequalities any further, but preferably continue with the search for completely different ones.

A very convenient fact on polytopes is that a linear function always attains its minimum (and, clearly, also its maximum) over a polytope in a vertex of it. This is simply due to the fact that the optimal solution points for a linear optimization problem defined on a polytope always constitute a face of the polytope. That is the reason, why in the previous section we just had to forget the minimizing element *for a while*, when we passed from minimizing a linear function over all incidence vectors to minimizing that linear function over their convex hull.

3. POLYTOPES ASSOCIATED WITH THE QAP

In our treatment of the QAP we will use a formulation of the problem in terms of certain graphs — mainly, because this provides us with some convenient ways to talk about the problem. Since in many application-driven instances (in particular, this holds for several of the instances in the QAPLIB, a commonly used set of test instances compiled by Burkard et al., 1997) the number of objects to be assigned might be smaller than the number of locations that are available, we will use a model that deals with n locations and m objects (with $m \leq n$). Not surprisingly, this turns out to be much more efficient than introducing $n-m$ "dummy-objects" that do not have any interaction with anybody.

From now on, let $\mathcal{M} = \{1, \ldots, m\}$ be the set of objects and $\mathcal{N} = \{1, \ldots, n\}$ the set of locations (with $m \leq n$). Let us denote by $\mathcal{X}^{m \times n} \in \{0,1\}^{m \times n}$ the set of all 0/1-matrices of size $m \times n$ that have *precisely* one "1" in every row and *at most* one "1" in every column (corresponding to the fact that every object has to be assigned to precisely one location, and every location can receive at most one object). Then the QAP can be formulated (see (L) in Section 1.) as the task to find $X^\star = (x^\star_{ij}) \in \mathcal{X}^{m \times n}$ such that

$$\sum_{i,k=1}^{m} \sum_{j,l=1}^{n} q_{ijkl} x^\star_{ij} x^\star_{kl} + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x^\star_{ij}$$

becomes minimal, where $q_{ijkl} \in \mathbb{R}$ are the *quadratic costs* (we may — and will — assume $q_{ijij} = 0$) and $c_{ij} \in \mathbb{R}$ are the *linear costs.*

Let $\mathcal{G}_{m,n} = (\mathcal{V}_{m,n}, \mathcal{E}_{m,n})$ be the graph with nodes $\mathcal{V}_{m,n} = \mathcal{M} \times \mathcal{N}$ and edges

$$\mathcal{E}_{m,n} = \left\{ \{(i,j),(k,l)\} \in \binom{\mathcal{V}_{m,n}}{2} : i \neq k, j \neq l \right\} \quad ,$$

where we denote by $\binom{M}{k}$ the set of subsets of a set M that have cardinality k. We usually denote an edge $\{(i,j),(k,l)\}$ by $[i,j,k,l]$ (notice that this implies $[i,j,k,l] = [k,l,i,j]$). The sets $\text{row}_i = \{(i,j) : 1 \leq j \leq n\}$ and $\text{col}_j = \{(i,j) : 1 \leq i \leq m\}$ are called the *i-th row* and the *j-th column* of $\mathcal{V}_{m,n}$, respectively. Thus, the graph $\mathcal{G}_{m,n}$ has all possible edges but the ones that have either both end points in the same row or in the same column.

By construction, the set $\mathcal{X}^{m \times n}$ of feasible solutions to our formulation of the QAP is in one-to-one correspondence with the m-cliques in $\mathcal{G}_{m,n}$. Now we put weights $c \in \mathbb{R}^{\mathcal{V}_{m,n}}$ and $d \in \mathbb{R}^{\mathcal{E}_{m,n}}$ on the nodes and edges, respectively, by letting $c_{(i,j)} = c_{ij}$ for every node $(i,j) \in \mathcal{V}_{m,n}$ and $d_{[i,j,k,l]} = q_{ijkl} + q_{klij}$ for every edge $[i,j,k,l] \in \mathcal{E}_{m,n}$. Then the QAP with linear costs c_{ij} and quadratic costs q_{ijkl} is equivalent to finding a cheapest node- and edge-weighted m-clique in the graph $\mathcal{G}_{m,n}$ weighted by (c,d).

This formulates the QAP as a *(linear) combinatorial optimization problem* in the sense of Section 2. (actually, it is still the same formulation as (L) in Section 1.). For a subset $W \subseteq \mathcal{V}_{m,n}$ of nodes of $\mathcal{G}_{m,n}$ let $\mathcal{E}_{m,n}(W)$ be the set of all edges of $\mathcal{G}_{m,n}$ that have *both* end points in W. We denote by $x^W \in \{0,1\}^{\mathcal{V}_{m,n}}$ the characteristic vector of W (i.e., $x^W_v = 1$ if and only if $v \in W$) and by $y^W \in \{0,1\}^{\mathcal{E}_{m,n}}$ the characteristic vector of $\mathcal{E}_{m,n}(W)$ (i.e., $y^W_e = 1$ if and only if $e \in \mathcal{E}_{m,n}(W)$). We call

$$\begin{aligned} \mathcal{QAP}_{m,n} &= \text{conv}\left\{ \left(x^C, y^C\right) : C \subseteq \mathcal{V}_{m,n} \ m\text{-clique in } \mathcal{G}_{m,n} \right\} \\ &\subseteq \mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\mathcal{E}_{m,n}} \end{aligned}$$

the *QAP-polytope* (for QAPs with m objects and n locations). Notice that we have $\dim(\mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\mathcal{E}_{m,n}}) = mn + \frac{mn(m-1)(n-1)}{2}$.

Now recall the Koopmans & Beckmann type instances. There quite often the flows (a_{ik}) or the distances (b_{jl}) are *symmetric* (meaning that $a_{ik} = a_{ki}$ holds for all pairs of objects i, k or that $b_{jl} = b_{lj}$ holds for all pairs of locations, respectively). In either case, this implies that

$$q_{ijkl} + q_{klij} = a_{ik}b_{jl} + a_{ki}b_{lj} = q_{ilkj} + q_{kjil}$$

holds for all quadratic costs, yielding

$$d_{[i,j,k,l]} = d_{[i,l,k,j]}$$

for all pairs $[i,j,k,l]$, $[i,l,k,j]$ of edges in our graph formulation. Since furthermore no clique in $\mathcal{G}_{m,n}$ can contain both edges $[i,j,k,l]$ and $[i,l,k,j]$, this suggests that in case of a *symmetric instance* (i.e., $q_{ijkl} + q_{klij} = q_{ilkj} + q_{kjil}$ holds for all quadratic costs) we can identify each pair $y_{[i,j,k,l]}$, $y_{[i,l,k,j]}$ of variables, which will reduce the number of variables by roughly 50%.

In order to formalize this, we define a hypergraph $\hat{\mathcal{G}}_{m,n} = (\mathcal{V}_{m,n}, \hat{\mathcal{E}}_{m,n})$ with the same set of nodes as our original graph $\mathcal{G}_{m,n}$ has and with hyperedges

$$\hat{\mathcal{E}}_{m,n} = \left\{ \{(i,j),(k,l),(i,l),(k,j)\} \in \binom{\mathcal{V}_{m,n}}{4} : i \neq k, j \neq l \right\} \ .$$

A hyperedge $\{(i,j),(k,l),(i,l),(k,j)\} \in \hat{\mathcal{E}}_{m,n}$ is denoted by $\langle i,j,k,l \rangle$. Thus, any hyperedge is the union

$$\langle i,j,k,l \rangle = [i,j,k,l] \cup [i,l,k,j]$$

of two edges from $\mathcal{G}_{m,n}$. We call two edges whose union gives a hyperedge *mates* of each other. Notice that we have

$$\langle i,j,k,l \rangle = \langle k,l,i,j \rangle = \langle i,l,k,j \rangle = \langle k,j,i,l \rangle \ .$$

For an edge $e \in \mathcal{E}_{m,n}$ we denote the hyperedge belonging to e (and to its mate) by $\text{hyp}(e)$. For any subset $W \subseteq \mathcal{V}_{m,n}$ define $\hat{\mathcal{E}}_{m,n}(W) = \{\text{hyp}(e) : e \in \mathcal{E}_{m,n}(W)\}$. Thus, if $C \subseteq \mathcal{V}_{m,n}$ is a clique of $\mathcal{G}_{m,n}$ (which we will also call a clique of $\hat{\mathcal{G}}_{m,n}$) then $\hat{\mathcal{E}}_{m,n}(C)$ consists of all hyperedges in $\hat{\mathcal{G}}_{m,n}$ that contain two nodes of C (since C is a clique, these two nodes then must precisely be the intersection of that hyperedge with C, and they must be "diagonal" in the "rectangle" formed by the end nodes of e and its mate). Finally, we denote for every subset $F \subseteq \hat{\mathcal{E}}_{m,n}$ of hyperedges the characteristic vector of F by z^F.

Now, we define the geometric structure that is especially suitable for symmetric QAPs, the *symmetric QAP-polytope*:

$$\mathcal{SQAP}_{m,n} = \text{conv}\left\{\left(x^C, z^C\right) : C \subseteq \mathcal{V}_{m,n}\ m\text{-clique in } \hat{\mathcal{G}}_{m,n}\right\}$$
$$\subseteq \mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}$$

We have $\dim(\mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}) = mn + \frac{mn(m-1)(n-1)}{4}$.

Next we will give integer programming formulations for both the general and the symmetric model. Recall that for a vector $v \in \mathbb{R}^I$ (I some finite set) and some subset $J \subseteq I$ we denote $v(J) = \sum_{j \in J} v_j$. We will need some further notations. For $i,k \in \mathcal{M}$ and $j \in \mathcal{N}$ let $((i,j) : \text{row}_k) = \{[i,j,k,l] : l \in \mathcal{N} \setminus \{j\}\}$ be the set of all edges connecting (i,j) with the k-th row, and let $\Delta^{(i,j)}_{(k,j)} = \{\langle i,j,k,l\rangle : l \in \mathcal{N} \setminus \{j\}\}$ be the set of all hyperedges containing both nodes (i,j) and (k,j). The proofs of the following theorems can be found in Kaibel, 1998 (for the symmetric version) as well as in Kaibel, 1997 (for both versions). Figure 1.2 illustrates equations (1.4) and (1.9). We usually draw hyperedges just by drawing the two corresponding edges. Notice that in our drawings a solid line or a solid disc will always indicate a coefficient 1, while a dashed line or a gray disc stand for a coefficient -1.

Theorem 1 *Let* $1 \le m \le n$.

(i) A vector $(x,y) \in \mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\mathcal{E}_{m,n}}$ *is a vertex of* $\mathcal{QAP}_{m,n}$*, i.e., the characteristic vector of an* m*-clique of* $\mathcal{G}_{m,n}$*, if and only if it satisfies the following conditions:*

(1.2)	$x(\text{row}_i) = 1$	$(i \in \mathcal{M})$
(1.3)	$x(\text{col}_j) \le 1$	$(j \in \mathcal{N})$
(1.4)	$-x_{(i,j)} + y\left((i,j) : \text{row}_k\right) = 0$	$(i,k \in \mathcal{M}, i \ne k, j \in \mathcal{N})$
(1.5)	$y_e \ge 0$	$(e \in \mathcal{E}_{m,n})$
(1.6)	$x_v \in \{0,1\}$	$(v \in \mathcal{V}_{m,n})$

(ii) A vector $(x,z) \in \mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}$ *is a vertex of* $\mathcal{SQAP}_{m,n}$*, i.e., the characteristic vector of an* m*-clique of* $\hat{\mathcal{G}}_{m,n}$*, if and only if it*

satisfies the following conditions:

(1.7) $x(\text{row}_i) = 1 \quad (i \in \mathcal{M})$

(1.8) $x(\text{col}_j) \leq 1 \quad (j \in \mathcal{N})$

(1.9)
$$-x_{(i,j)} - x_{(k,j)} + z\left(\Delta^{(i,j)}_{(k,j)}\right) = 0 \quad (i,k \in \mathcal{M}, i < k, j \in \mathcal{N})$$

(1.10) $z_h \geq 0 \quad (h \in \hat{\mathcal{E}}_{m,n})$

(1.11) $x_v \in \{0,1\} \quad (v \in \mathcal{V}_{m,n})$

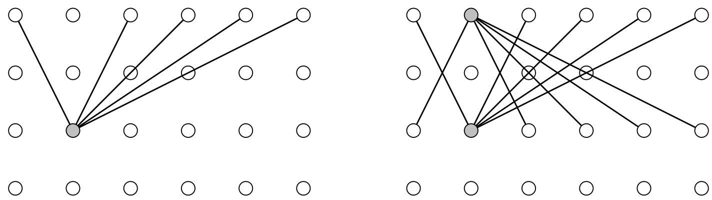

Figure 1.2 Equations (1.4) and (1.9).

Once one starts to investigate $\mathcal{QAP}_{m,n}$ and $\mathcal{SQAP}_{m,n}$ closer, it turns out very soon that the structures of the polytopes are quite different for the cases $m \leq n-2$ and $m \in \{n-1, n\}$. In fact, the case $m \leq n-2$ is much nicer to handle than the other two. Here, we just want to give a result that shows that actually the case $m = n-1$ reduces to the case $m = n$, thus leaving us with only one "less convenient case". Again the proof of the theorem can be found in Kaibel, 1997 and Kaibel, 1998.

Theorem 2 *For $n \geq 2$, $\mathcal{QAP}_{n-1,n}$ is affinely isomorphic to $\mathcal{QAP}_{n,n}$, and $\mathcal{SQAP}_{n-1,n}$ is affinely isomorphic to $\mathcal{SQAP}_{n,n}$.*

The affine maps from $\mathbb{R}^{\mathcal{V}_{n,n}} \times \mathbb{R}^{\mathcal{E}_{n,n}}$ to $\mathbb{R}^{\mathcal{V}_{n-1,n}} \times \mathbb{R}^{\mathcal{E}_{n-1,n}}$ and from $\mathbb{R}^{\mathcal{V}_{n,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{n,n}}$ to $\mathbb{R}^{\mathcal{V}_{n-1,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{n-1,n}}$ giving the isomorphisms in the theorem are quite simple. They are just the canonical projections that "forget" all coordinates belonging to nodes in row_n or to (hyper)edges that intersect row_n.

We conclude this section by some results on $\mathcal{QAP}_{m,n}$ and $\mathcal{SQAP}_{m,n}$ which can be found in a more detailed version (in particular including proofs) in Kaibel, 1997, Jünger and Kaibel, 1997b, Jünger and Kaibel, 1996, and Kaibel, 1998.

It is rather obvious that for $m' \leq m \leq n$ the polytopes $\mathcal{QAP}_{m',n}$ and $\mathcal{SQAP}_{m',n}$ arise as "canonical projections" of the polytopes $\mathcal{QAP}_{m,n}$ and $\mathcal{SQAP}_{m,n}$, respectively. This implies that whenever an inequality is valid (facet-defining) for $\mathcal{QAP}_{m,n}$ or $\mathcal{SQAP}_{m,n}$ and it has all its nonzero coefficients within the first m' rows, then the corresponding inequality is valid (facet-defining) for $\mathcal{QAP}_{m',n}$ or $\mathcal{SQAP}_{m',n}$, respectively.

But also $\mathcal{QAP}_{m,n}$ and $\mathcal{SQAP}_{m,n}$ have a relationship; the latter is an affine image (not under an isomorphism) of the first. In particular, if an inequality is valid for $\mathcal{QAP}_{m,n}$ and it is *symmetric* in the sense that each two edges $[i,j,k,l]$, $[i,l,k,j]$ have the same coefficient, then the inequality induces in the obvious way a valid inequality for $\mathcal{SQAP}_{m,n}$. Let us call a face of $\mathcal{QAP}_{m,n}$ *symmetric* if it *can* be defined by a symmetric inequality. Then it turns out that a symmetric face F of $\mathcal{QAP}_{m,n}$ (which induces a face o $\mathcal{SQAP}_{m,n}$, as indicated above) actually induces a facet of $\mathcal{SQAP}_{m,n}$ if and only if all faces of $\mathcal{QAP}_{m,n}$ that properly contain F are *not* symmetric. More generally, the face lattice of $\mathcal{SQAP}_{m,n}$ arises from the face lattice of $\mathcal{QAP}_{m,n}$ by deleting all faces that are not symmetric.

These various connections between the different polytopes we have to consider give several possibilities to carry over results between them. However, there are also interesting connections to polytopes "outside the QAP world". In particular, it turns out that $\mathcal{QAP}_{m,n}$ can be viewed as a certain face of a cut polytope. This rises hope that one might take profit from the rich knowledge on cut polytopes (see, e.g., Deza and Laurent, 1997) for the investigations of the QAP-polytopes. In fact, this turns out to be true, as we will see in Section 5..

Another connection to different kinds of polytopes is the following. It turns out that several well-known polytopes like the *traveling salesman polytope* or the *linear ordering polytope* can be obtained as quite simple projections (just "forgetting coordinates") of $\mathcal{QAP}_{n,n}$. This phenomenon corresponds to the possibility to obtain problems like the traveling salesman problem or the linear ordering problem as "immediate special cases" of the QAP (which explains to some extent the astonishing resistance of the QAP against all attacks to practically solve it).

4. THE STAR-TRANSFORMATION

Having defined all these polytopes as convex hulls of certain characteristic vectors, according to the principles of polyhedral combinatorics explained in Section 2. the next (and crucial) step now is to find (partial) linear descriptions of the polytopes. That means, we aim for finding systems of linear equations whose solution spaces are the affine hulls of the polytopes as well as linear inequalities that are valid for the polytopes and, preferably, even define facets of them. While showing validity of equations and inequalities will always be done by ad-hoc arguments in our case, the proofs that the equation systems that we propose indeed completely describe the affine hulls or that some inequalities indeed define facets (i.e., faces of largest possible dimension) require some more elaborate techniques.

In both cases (equation systems and inequalities) we have to solve tasks of the following type: given $X \subseteq \{0,1\}^s$ and a system $Ax = b$ of linear equations that are valid for X (with $A \in \mathbb{Q}^{r \times s}$ having full row rank and $b \in \mathbb{Q}^r$) prove that $\mathrm{aff}(X) = \{x \in \mathbb{Q}^s : Ax = b\}$ holds.

In order to prove this, we proceed as follows. First, we identify a subset $\mathcal{B} \subseteq \{1, \ldots, s\}$ of variable indices with $|\mathcal{B}| = r$ such that the columns of A that correspond to $\mathcal{B}$ (called a *basis* of A) form a non-singular matrix. Then, by a dimension argument, it suffices to show

$$\mathrm{aff}\big(\{e_i : i \in \mathcal{B}\} \cup X\big) = \mathbb{Q}^s \ .$$

Our way to prove this equation then is to construct all unit vectors $e_1, \ldots, e_s \in \mathbb{Q}^s$ as linear combinations of $\{e_i : i \in \mathcal{B}\} \cup X$.

The crucial task in the proofs thus is to construct suitable linear combinations of certain vertices, i.e., of characteristic vectors of certain feasible solutions. If one starts to play around with vertices of the polytopes we have introduced in Section 3., one soon will find that this is quite inconvenient, because it turns out to be very hard to obtain vectors with a small and well-structured *support* (i.e., set of indices of nonzero components).

However, this is not a problem that is inherent to the geometry of the polytopes or even to the QAP itself. It is only a matter of the actual coordinate representation that we have chosen for modeling the problem. And indeed, it turns out that one can find different coordinate representations of the QAP-polytopes (in different, lower dimensional, ambient spaces) that make it much more comfortable to build linear combinations from the vertices.

The key is the observation that the last column (or any single column) in the (hyper-)graph model contains redundant information on the feasible solutions: the intersection of an m-clique $C \subseteq \mathcal{V}_{m,n}$ with the first $n-1$ columns already determines the intersection of C with the last column. In case of $m = n$ even the last column *and* the last row (or any single column *and* any single row) are redundant in this sense. This suggests to remove the last column (and the last row in case of $m = n$) from the model. If we remove the last column an m-clique $C \subseteq \mathcal{V}_{m,n}$ may become either an $(m-1)$- or an m-clique in $\mathcal{V}_{m,n-1}$

(depending on whether C intersects the last column or not). If $m = n$ holds and we remove both the last column and the last row, then an n-Clique $C \subseteq \mathcal{V}_{n,n}$ becomes an $(n-1)$- or an $(n-2)$-clique in $\mathcal{V}_{n-1,n-1}$ (depending on whether (n, n) is contained in C or not).

For the QAP-polytopes, removing nodes and (hyper-)edges corresponds to orthogonal projections that "forget" the coordinates belonging to the nodes and (hyper-) edges that are removed. Since we just want to find different coordinate representations of the polytopes (with their geometry unchanged), we must ensure that these projections actually give affine isomorphisms of the polytopes.

For $m \leq n^\star$ define the polytopes

$$\mathcal{QAP}^\star_{m,n^\star} =$$

$$\operatorname{conv}\left\{ \left(x^C, y^C\right) : C \subseteq \mathcal{V}_{m,n^\star} \ m\text{- or } (m-1)\text{-clique in } \mathcal{G}_{m,n^\star} \right\}$$

and

$$\mathcal{SQAP}^\star_{m,n^\star} =$$

$$\operatorname{conv}\left\{ \left(x^C, z^C\right) : C \subseteq \mathcal{V}_{m,n^\star} \ m\text{- or } (m-1)\text{-clique in } \hat{\mathcal{G}}_{m,n^\star} \right\} ,$$

which are objects in $\mathbb{R}^{\mathcal{V}_{m,n^\star}} \times \mathbb{R}^{\mathcal{E}_{m,n^\star}}$ and $\mathbb{R}^{\mathcal{V}_{m,n^\star}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n^\star}}$, respectively. The following result was shown (for the different versions of QAP-polytopes) in Jünger and Kaibel, 1997b, Jünger and Kaibel, 1996, and Kaibel, 1997.

Theorem 3 *Let* $3 \leq m \leq n$.

(i) The polytopes $\mathcal{QAP}_{m,n}$ *and* $\mathcal{SQAP}_{m,n}$ *are affinely isomorphic to the polytopes* $\mathcal{QAP}^\star_{m,n-1}$ *and* $\mathcal{SQAP}^\star_{m,n-1}$*, respectively.*

(i) The polytopes $\mathcal{QAP}_{n,n}$ *and* $\mathcal{SQAP}_{n,n}$ *are affinely isomorphic to the polytopes* $\mathcal{QAP}^\star_{n-1,n-1}$ *and* $\mathcal{SQAP}^\star_{n-1,n-1}$*, respectively.*

The change of coordinate representation described in Theorem 3 is called the *star-transformation*. What have we gained by it? As desired, we have gained the possibility to obtain easily such simple vectors as the ones shown in Figure 4. as linear combinations of vertices. And in fact, these types of vectors are the basic ingredients for the proofs of all results coming up in Section 5..

Another advantage of applying the star-transformation to the QAP-polytopes is that the "dimensional gaps" between their dimensions and the dimensions of the spaces they are located in has become much smaller. This leads to much smaller systems of equations describing the affine hulls. In fact, it turns out that the affine hulls of $\mathcal{QAP}^\star_{m,n^\star}$

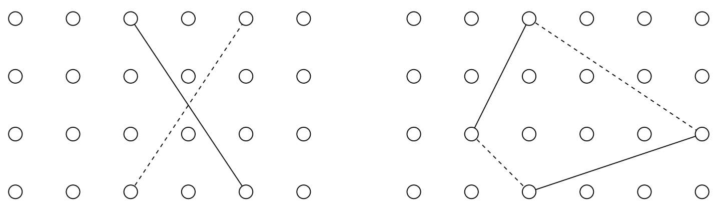

Figure 1.3 Some very simple vectors that can easily be obtained as linear combinations of vertices of $\mathcal{QAP}^{\star}_{m,n^{\star}}$.

and $\mathcal{SQAP}^{\star}_{m,n^{\star}}$ can be described by very simple equation systems, which in particular have very convenient bases (see Jünger and Kaibel, 1997b, Jünger and Kaibel, 1996, and Kaibel, 1997).

5. FACIAL DESCRIPTIONS OF QAP-POLYTOPES

Let us first consider the affine hulls of $\mathcal{QAP}_{m,n}$ and $\mathcal{SQAP}_{m,n}$, i.e., the (linear) equations holding for these polytopes. In case of $m \leq n-2$, it turns out that the equations in the integer programming formulations of Theorem 1 do already suffice.

Theorem 4 *Let* $1 \leq m \leq n-2$.

(i) The affine hull $\operatorname{aff}(\mathcal{QAP}_{m,n})$ *is the set of solutions of the equations* (1.2) *and* (1.4). *If one removes for every pair* $i, k \in \mathcal{M}$ *one of the equations in* (1.4) *then one obtains an irredundant system of equations describing* $\operatorname{aff}(\mathcal{QAP}_{m,n})$. *In particular, we have:*

$$\dim(\mathcal{QAP}_{m,n}) = \dim(\mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\mathcal{E}_{m,n}}) - \left(m^2 n - mn - \frac{1}{2}m^2 + \frac{3}{2}m\right)$$

(ii) The affine hull $\operatorname{aff}(\mathcal{SQAP}_{m,n})$ *is the set of solutions of the equations* (1.7) *and* (1.9). *Here, in the symmetric case, the equations already form an irredundant system. In particular, we obtain:*

$$\dim(\mathcal{SQAP}_{m,n}) = \dim(\mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}) - \left(\frac{1}{2}m^2 n - \frac{1}{2}mn + m\right)$$

The proofs of both parts of the theorem can be found in Kaibel, 1997.

Now let us turn to the case $m = n$. Here, one immediately finds that the inequalities (1.3) and (1.8) ("$x(\mathrm{col}_j) \leq 1$") actually must be satisfied with equality by the vertices of the polytopes $\mathcal{QAP}_{n,n}$ and $\mathcal{SQAP}_{n,n}$. Moreover, of course also the "column versions" of the equations (1.4) and (1.9) are satisfied by all vertices of the respective polytopes. In fact, it turns out that by these observations we have collected enough equations to describe the affine hulls of $\mathcal{QAP}_{n,n}$ and $\mathcal{SQAP}_{n,n}$.

Theorem 5 *Let $n \geq 1$.*

(i) The affine hull $\mathrm{aff}(\mathcal{QAP}_{n,n})$ *is the set of solutions of the equations* (1.2), (1.4), *and*

$$x(\mathrm{col}_j) = 1 \qquad (j \in \mathcal{N}) \tag{1.12}$$

$$-x_{(i,j)} + y\left((i,j) : \mathrm{col}_l\right) = 0 \qquad (i,j,l \in \mathcal{N}, j \neq l) \ . \tag{1.13}$$

We have

$$\dim(\mathcal{QAP}_{n,n}) = \dim(\mathbb{R}^{\mathcal{V}_{n,n}} \times \mathbb{R}^{\mathcal{E}_{n,n}}) - \left(2n^3 - 5n^2 + 5n - 2\right) \ .$$

(ii) The affine hull $\mathrm{aff}(\mathcal{SQAP}_{n,n})$ *is the set of solutions of the equations* (1.7), (1.9), *and*

$$x(\mathrm{col}_j) = 1 \qquad (j \in \mathcal{N}) \tag{1.14}$$

$$-x_{(i,j)} - x_{(i,l)} + z\left(\Delta^{(i,j)}_{(i,l)}\right) = 0 \qquad (i,j,l \in \mathcal{N}, j < l) \ . \tag{1.15}$$

We have

$$\dim(\mathcal{SQAP}_{n,n}) = \dim(\mathbb{R}^{\mathcal{V}_{n,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{n,n}}) - \left(n^3 - 2n^2 + 2n - 1\right) \ .$$

The result of Part (i) was also proved by Rijal, 1995 and Padberg and Rijal, 1996. The dimension of $\mathcal{QAP}_{n,n}$ (without an explicit system of equations) was already computed by Barvinok, 1992 in his investigations of the connections between polytopes coming from combinatorial optimization problems and the representation theory of the symmetric group. For a proof in our notational setting see Jünger and Kaibel, 1997b, where one may also find information which equations have to be removed in order to obtain an irredundant system. The proof of the symmetric version (Part (ii)) is in Jünger and Kaibel, 1996. Unlike with the case $m \leq n-2$, the system given in the theorem is redundant also in the symmetric case. How to obtain an irredundant system can be found in Kaibel, 1997. Notice that Padberg and Rijal, 1996 already conjectured Part (ii).

Now that we know everything about the equations that are valid for $\mathcal{QAP}_{m,n}$ and $\mathcal{SQAP}_{m,n}$, let us turn to inequalities. We start with the "trivial inequalities" (i.e., the bounds on the variables). If we state that some inequality is "implied" by some others then this will always mean that *all* solutions (not only the integral ones) to the other inequalities satisfy also the inequality under inspection.

Theorem 6 *Let* $3 \leq m \leq n$.

(i) The inequalities

$$y_e \geq 0 \qquad (e \in \mathcal{E}_{m,n})$$

define facets of $\mathcal{QAP}_{m,n}$.

(ii) The inequalities

$$\begin{aligned} x_v &\geq 0 & \qquad (v \in \mathcal{V}_{m,n}) \\ x_v &\leq 1 & \qquad (v \in \mathcal{V}_{m,n}) \\ y_e &\leq 1 & \qquad (e \in \mathcal{E}_{m,n}) \end{aligned}$$

are implied by the equations (1.2), (1.4), *and the nonnegativity constraints* $y \geq 0$ *on the edge variables.*

(iii) The inequalities

$$\begin{aligned} x_v &\geq 0 & \qquad (v \in \mathcal{V}_{m,n}) \\ z_h &\geq 0 & \qquad (h \in \hat{\mathcal{E}}_{m,n}) \end{aligned}$$

define facets of $\mathcal{SQAP}_{m,n}$.

(iv) The inequalities

$$\begin{aligned} x_v &\leq 1 & \qquad (v \in \mathcal{V}_{m,n}) \\ z_h &\leq 1 & \qquad (h \in \hat{\mathcal{E}}_{m,n}) \end{aligned}$$

are implied by the equations (1.7), (1.9), *and the nonnegativity constraints* $x \geq 0$ *and* $z \geq 0$ *on the node and hyperedge variables.*

The proof of the theorem can be found in Kaibel, 1997. The case $m = n$ is also in Jünger and Kaibel, 1997b and Jünger and Kaibel, 1996. Parts (i) and (ii) for the case $m = n$ have also been proved by Rijal, 1995 and Padberg and Rijal, 1996.

Let us now consider other inequalities for $m \leq n-2$. Although in this case the "column versions" (1.13) and (1.15) do not hold for $\mathcal{QAP}_{m,n}$ and $\mathcal{SQAP}_{m,n}$, respectively, the corresponding inequalities

$$(1.16) \qquad -x_{(i,j)} + y\left((i,j) : \mathrm{col}_l\right) \leq 0 \qquad (i \in \mathcal{M}, j, l \in \mathcal{N}, j \neq l)$$

and

$$(1.17) \qquad -x_{(i,j)} - x_{(i,l)} + z\left(\Delta^{(i,j)}_{(i,l)}\right) \leq 0 \qquad (i \in \mathcal{M}, j, l \in \mathcal{N}, j < l)$$

are valid.

There is one more interesting class of inequalities that in case of $m = n$ are valid as equations (and thus implied by the equations we have described above). For $j, l \in \mathcal{N}$ $(j \neq l)$ let us denote by $(\mathrm{col}_j : \mathrm{col}_l)$ the set of all edges connecting column j with column l, and let $\langle \mathrm{col}_j : \mathrm{col}_l \rangle = \{\mathrm{hyp}(e) : e \in (\mathrm{col}_j : \mathrm{col}_l)\}$ be the set of hyperedges connecting column j with column l. Then the inequalities

$$(1.18) \qquad x\left(\mathrm{col}_j \cup \mathrm{col}_l\right) - y\left(\mathrm{col}_j : \mathrm{col}_l\right) \leq 1 \qquad (j, l \in \mathcal{N}, j < l)$$

and

$$(1.19) \qquad x\left(\mathrm{col}_j \cup \mathrm{col}_l\right) - z\left(\langle \mathrm{col}_j : \mathrm{col}_l \rangle\right) \leq 1 \qquad (j, l \in \mathcal{N}, j < l)$$

are valid for $\mathcal{QAP}_{m,n}$ and $\mathcal{SQAP}_{m,n}$, respectively.

Theorem 7 *Let* $4 \leq m \leq n-2$.

(i) The inequalities (1.16) *and* (1.18) *define facets of* $\mathcal{QAP}_{m,n}$.

(ii) The inequalities (1.3) *are implied by the inequalities* (1.18) *and the equations* (1.2) *and* (1.4).

(iii) The inequalities (1.17) *and* (1.19) *define facets of* $\mathcal{SQAP}_{m,n}$.

(iv) The inequalities (1.8) *are implied by the inequalities* (1.19) *and the equations* (1.7) *and* (1.9).

The proofs of all parts of the theorem are in Kaibel, 1997.

The final class of inequalities that we will consider is the class of *box-inequalities*. This was the first large (i.e., exponentially large) class of facet-defining inequalities discovered for the QAP-polytopes (and it is still the only one that is known). A large part of its importance, however, is not due to this theoretical property, but due to the fact, that using some of these inequalities as cutting planes one can indeed significantly improve the lower bounds obtained by classical LP-based bounding procedures — in several cases even up to the possibility to compute optimal solutions by a pure cutting plane algorithm (see Section 6.).

The starting point is the following trivial observation: if $\gamma \in \mathbb{Z}$ is an integer number then $\gamma(\gamma - 1) \geq 0$ must hold. Now suppose that $S, V \subseteq \mathcal{V}_{m,n}$ are disjoint subsets of nodes and that $\beta \in \mathbb{Z}$ is an arbitrary integer. Let $(x, y) \in \mathcal{QAP}_{m,n}$ be a vertex of $\mathcal{QAP}_{m,n}$. By the above observation the quadratic inequality

$$(1.20) \qquad \big(x(T) - x(S) - \beta\big)\big(x(T) - x(S) - (\beta - 1)\big) \geq 0$$

holds. But since (x, y) is the characteristic vector of the nodes and edges in some m-clique $C \subseteq \mathcal{V}_{m,n}$ of $\mathcal{G}_{m,n}$ we have $x(S)x(T) = y(S : T)$ and $x(R)x(R) = x(R) + 2y(R)$ for every $R \subseteq \mathcal{V}_{m,n}$ (with $y(R) = y(\mathcal{E}_{m,n}(R))$, which allows us to rewrite the quadratic inequality into the following linear *ST-inequality*:

$$(1.21) \quad -\beta x(S) + (\beta - 1)x(T) - y(S) - y(T) + y(S : T) \leq \frac{\beta(\beta - 1)}{2}$$

It is obvious from (1.20) that the vertices of the face of $\mathcal{QAP}_{m,n}$ defined by this ST-inequality are precisely the characteristic vectors of those m-cliques $C \subseteq \mathcal{V}_{m,n}$ which satisfy

$$|C \cap T| - |C \cap S| \in \{\beta, \beta - 1\} \ .$$

As mentioned in Section 3., $\mathcal{QAP}_{m,n}$ can be viewed as a certain face of the cut-polytope (associated with the complete graph on $mn+1$ nodes). We just mention that the ST-inequalities correspond to certain *hypermetric inequalities* for the cut-polytope. For details on this connection we refer to Jünger and Kaibel, 1997a and Kaibel, 1997, and for answers to nearly every question on the cut-polytope (like "what are hypermetric inequalities?") and related topics to Deza and Laurent, 1997.

In the light of the remarks at the end of Section 3. the symmetric ones among the ST-inequalities are of special interest. Here is a possibility to choose S and T such that the resulting inequality is symmetric. Let $P_1, P_2 \subseteq \mathcal{M}$ and $Q_1, Q_2 \subseteq \mathcal{N}$ with $P_1 \cap P_2 = \varnothing$ and $Q_1 \cap Q_2 = \varnothing$ and take $S = (P_1 \times Q_1) \cup (P_2 \times Q_2)$ as well as $T = (P_1 \times Q_2) \cup (P_2 \times Q_1)$. An ST-inequality arising from sets S and T of this type is called a *4-box-inequality* (see Figure 1.4). In Jünger and Kaibel, 1997a the following result is proved.

Theorem 8 *An ST-inequality is symmetric if and only if it is a 4-box-inequality.*

If one of the sets P_1, P_2, Q_1, or Q_2 is empty, then we call the corresponding 4-box-inequality a *2-box-inequality.* If P_1 or P_2 is empty and Q_1 or Q_2 is empty, the 4-box-inequality is a *1-box-inequality.* While theoretical investigations of the whole class of 4-box-inequalities seem to be too difficult, the 2-box-inequalities are studied extensively in Kaibel, 1997. In particular, the facet-defining ones among them are identified;

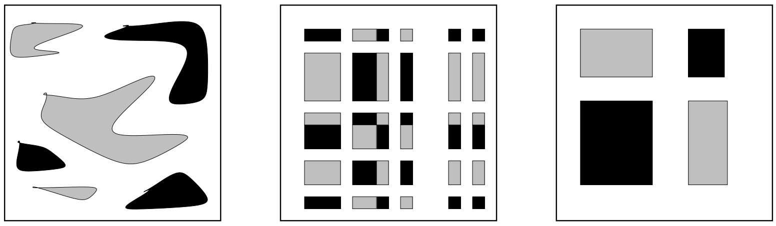

Figure 1.4 The node sets of ST-inequalities in general, of 4-box inequalities, and of 4-box inequalities after suitable permutations of rows and columns. The set S is always indicated by the gray parts, the set T by the black ones.

it turns out that most of them define facets of $\mathcal{QAP}_{m,n}$. Rather than stating the result in its whole generality, we prefer to give a theorem that shows a large class of 1-box-inequalities that are facet-defining for $\mathcal{QAP}_{m,n}$ (and thus, for $\mathcal{SQAP}_{m,n}$). In particular, this class contains the inequalities that are used in Section 6. for computing lower bounds. The proof of the theorem is contained in Jünger and Kaibel, 1997a.

Theorem 9 *Let $7 \leq m \leq n$. Let $P \subseteq \mathcal{M}$ and $Q \subseteq \mathcal{N}$ generate $T = P \times Q \subseteq \mathcal{V}_{m,n}$, and let $\beta \in \mathbb{Z}$ be an integer number such that*

- $\beta \geq 2$,
- $|P|, |Q| \geq \beta + 2$,
- $|P|, |Q| \leq n - 3$, *and*
- $|P| + |Q| \leq n + \beta - 5$

hold. Then the 1-box inequality

$$(\beta - 1)x(\mathcal{T}) - y(T) \leq \frac{\beta(\beta - 1)}{2}$$

defines a facet of $\mathcal{QAP}_{m,n}$ and the corresponding 1-box-inequality

$$(\beta - 1)x(T) - z(T) \leq \frac{\beta(\beta - 1)}{2}$$

defines a facet of $\mathcal{SQAP}_{m,n}$.

Before closing this section, we want to point out that the proofs of the results presented here heavily rely on the techniques described in Section 4. and on the various connections between the different polytopes arising from projections mentioned in Section 3.. While these projections are very helpful for avoiding to do work twice, the proofs that have to be done from the scratch are extremely simplified by the star-transformations. Even to prove the statements on the dimensions and the trivial facets of the polytopes (at least in case of $m = n$) are very tedious without using the star-transformation (see Rijal, 1995). For the more complicated situations, in particular with the box-inequalities, the proofs seem to be impossible without exploiting the star-transformation — at least for the author.

6. CUTTING PLANE ALGORITHMS FOR QAPS

The insight into the geometry of the different QAP-polytopes described in the previous section can be used to compute lower bounds or even optimal solutions in the way explained in Section 2.. In Jünger and Kaibel, 1997a, Kaibel, 1998, and Kaibel, 1997 experiments with cutting plane algorithms exploiting the results on the facial structures of the QAP-polytopes are described. Here, we just report on some of the experimental results in order to show that the polyhedral investigations of the QAP indeed help to improve the algorithmic solvability of the problem.

The cutting plane code that we have implemented is suited for symmetric instances and can handle both, the $m = n$ as well as the $m \leq n-2$ case. In the first case, the initial LP consists of a complete equation system plus the nonnegativity constraints on the variables. In the second case, the initial LP contains, again, a complete system of equations, the nonnegativity constraints on the variables, and, additionally, the inequalities (1.17) and (1.19). Of course, one can transform every instance with $m \leq n - 2$ into an instance with $m = n$ by adding "dummy objects". In Kaibel, 1998 it is shown that for such instances the bounds obtained from the initial LPs with and without dummy objects coincide. The bound from the initial LP in case of $m = n$ is (empirically) slightly weaker than the corresponding bound in the non-symmetric model (see Jünger and Kaibel, 1996), where the latter LP is equivalent to the linear programming relaxation of (AJ) (see Section 1.).

The algorithm first sets up the initial LP, and then solves it to obtain a lower bound on the optimal value of the QAP. If the LP-solution happens to be an integer vector (i.e., a 0/1-vector in this case), then, by Theorem 1, it is the characteristic vector of an optimal solution to the problem. Otherwise, the algorithm tries to separate the (fractional) LP-solution from the feasible solutions by searching for 4-box inequalities that are violated by it. If such inequalities are found (which is not guaranteed), then they are added to the LP, a new (hopefully better) lower bound is computed, and the process is iterated for a specified number of rounds.

Actually, our *separation algorithm* is quite simple. Since some initial experiments showed that 1-box inequalities with $\beta \in \{2,3\}$ seem to be most valuable within the cutting plane algorithm, we have restricted to this type of inequalities in our experiments. Because there seems to be no obvious way to solve the separation problem (for this class of inequalities) fast and exactly (i.e., either to find violated inequalities or to affirm that no violated inequality exists within the class), we have implemented a rather simple heuristic for it. We just guess 1-box inequalities randomly and then try to increase their left-hand-sides (with respect to the current fractional solution) by changing the box in a 2-opt way. Actually, the experiments show that usually this primitive procedure detects quite a lot of violated inequalities. We then choose the most violated ones among them and add them to the LP. This way, about 0.2 to 0.4 times the number of initial constraints are added to the LP in every iteration of the cutting plane procedure. In order to control the size of the LP we also remove inequalities if they have been redundant (i.e., non-binding for the optimal LP-solution) for several cutting plane iterations in a row. For details on the algorithm we refer to Kaibel, 1997, Jünger and Kaibel, 1997a, and Kaibel, 1998.

The LPs have been solved by the barrier method of the CPLEX 4.0 package. Using the (primal or dual) simplex algorithm did not pay off at all, which is due to the very high primal and dual degeneracy of the LPs. This is very much in accordance with the computational experiments performed by Resende et al., 1995 with LPs that are equivalent to our initial LP in the non-symmetric $m = n$ case. All our experiments were carried out on a Silicon Graphics Power Challenge machine using the parallel version of the CPLEX barrier code on four processors.

Table 1.1 shows the results for all instances from the QAPLIB with $m = n \leq 20$ (they are all symmetric). For all these instances, optimal solutions are known (and published in the QAPLIB). The table gives the bounds obtained from the initial LP as well as the ones obtained by the cutting plane procedure. Notice that, since all objective functions are integral, we can round up every bound to the next integer number. The columns titled *qual* give the ratios of the respective bound and the optimal solution value. The running times for the cutting plane algorithm are specified in seconds. The column titled *iter* shows the number of cutting plane iterations, i.e., the number of LPs solved to obtain the bound.

The results show that the box-inequalities indeed are quite valuable for improving LP based lower bounds. For several instances the bounds even match the optimal solution values, and for most of the other instances, a large part of the gaps between the bounds obtained from the initial LPs and the optimal solution values is closed by adding box-inequalities.

While the quality of the bounds obtained from the cutting plane algorithm is quite good, for many instances the running times are rather large. We will address this point at the end of this section. But let us first turn to the experiments with instances where $m \leq n-2$ holds. Table 1.2 and 1.3 report on the results obtained by the cutting plane code on the `esc16`- and `esc32`-instances (with $n = 16$ and $n = 32$, respectively) from the QAPLIB, where here the columns titled *box* contain the bounds computed by the cutting plane code. Since for these instances both the flow- as well as the distance-matrix are symmetric and integral, the optimal solution value must be an even integer number. Thus we can round up every bound to the next even integer.

For the `esc16`-instances, the column titled *opt* contains the optimal solution values and the column titled *speed up* contains the quotients of the running times with and without dummy objects. All these instances were solved to optimality for the first time by Clausen and Perregaard, 1997, who used a Branch & Bound code running on a parallel machine with 16 i860 processors. The column titled *ClPer* shows the running times of their algorithm. The cutting plane code finds for all these instances (except for `esc16a`) the optimal solution value within (more or less) comparable running times.

For the `esc32`-instances the cutting plane code always produces the best known lower bounds. The column titled *upper* contains the values of the currently best known feasible solutions and the column titled *prev lb* contains the previously best known lower bounds. The instances `esc32e`, `esc32f`, and `esc32g` have been solved to optimality for the first time by a parallel Branch & Bound code of Brüngger et al., 1996. For the other instances (except for `esc32c`) the cutting plane algorithm improves the previously best known lower bounds, where the improvement for `esc32a` is the most significant one. The running times that have a "$\star$" in front are not measured exactly due to some problems with the queuing system of our machine.

While all these experiments show that the polyhedral investigations indeed pay off with respect to the goal of the computation of tight lower bounds, the running times of the cutting plane algorithm are (for most of the instances) quite large. In order to obtain a "practical" bounding procedure (that, in particular, might be incorporated into Branch

& Bound frameworks) the algorithm has to be speeded up significantly. One approach into this direction is to implement more elaborate separation strategies. But the potential of this kind of improvements is limited, since already the initial LPs become really large for larger values of m and n. For instance, for `esc32a` the initial LP has 149600 variables and 22553 equations.

One way to reduce the sizes of the LPs is to exploit the fact that quite often the objective functions are rather sparse. For example, in case of a Koopmans & Beckmann instance the flow matrix might be sparse because there are lots of pairs of objects which do not have any flows between them. Actually, this is true for many instances in the QAPLIB. For example, `els19` has a (symmetric) flow matrix, where out of the 171 pairs of objects only 56 have a nonzero flow.

In our graph model, a pair $i, k \in \mathcal{M}$ of objects without any flow between i and k has the effect that all (hyper-)edges connecting row i and row k have objective function coefficient zero. This means that one might "project out" all variables corresponding to these (hyper-)edges and solve the problem over the corresponding projected polytope. For `els19` this reduces the number of variables from 29,602 down to 9,937.

Of course, (unlike with the projections used for the star-transformation) in general the projection will change the geometric properties of the polytope. Thus, one has to do theoretical investigations of the projected polytopes depending on the *flow graph*, i.e., the graph defined on the objects and having an edge for every pair of objects which does have some flow. This was suggested already by Padberg and Rijal, 1996. First results can be found in Kaibel, 1997. Again, lots of results presented in Section 5. can be carried over by the observation that an inequality that is valid (facet-defining) for the unprojected polytope immediately yields a valid (facet-defining) inequality for the projected polytope as long as the inequality has no nonzero coefficient on a (hyper-)edge that belongs to a pair of objects which do not have any flow between them. In particular, for every clique in the flow graph there are a lot of box-inequalities which are also valid (facet-defining) for the projected polytopes.

In Elf, 1999 some computational experiments with a cutting plane algorithm working with the "sparse models" are performed. Table 1.4 shows results for the `esc32` instances.

Comparing these results with the ones in Table 1.3 one finds that (at least for the `esc32` instances) the running times of the cutting plane algorithm are reduced substantially by exploiting sparsity of the objective functions. While this might be paid by a weaker bound (`esc32b`, `esc32c`, `esc32d`, `esc32h`), it is also possible that the bound becomes better (`esc32a`). Notice that in the sparse model, it takes only about one minute to compute the optimal solution values of `esc32e` and `esc32f`.

7. CONCLUSION

We close with three important aspects of the polyhedral work on the QAP that we have surveyed in this chapter.

- The techniques that have been developed for theoretical investigations of QAP-polytopes, like the projections between the different types of polytopes and, especially important, the star-transformation, provide tools which make polyhedral studies on the QAP possible. In fact, they have led to first considerable insights into the polyhedral structure of the QAP, which before was one of the few problems among the classical combinatorial optimization problems about which we lacked any deeper polyhedral knowledge.

- The practical experiments with a cutting plane algorithm that exploits the polyhedral results show that this type of approach has a great potential for computing tight lower bounds and even optimal solutions. However, the tightness of the bounds is payed by considerable running times for larger instances.

- In order to overcome the relatively large running times one might exploit sparsity in the objective function. First theoretical and experimental studies have shown that at least special kinds of sparsity (coming from sparse flow structures on the objects) can be handled theoretically and can be used to improve the running times of cutting plane algorithms substantially.

These points suggest, in our opinion, the further lines of research in the area of polyhedral combinatorics of the QAP. The most promising possibility to really push the (exact) solvability of QAPs beyond the current limits by polyhedral methods is to extend the work on the sparse model. One direction here, of course, is the further investigation of the structures of the QAP-polytopes in the sparse model. In particular, one could search for facets of the polytopes that are not projections of facets of polytopes in the dense model. Another direction is to study models that do not only exploit sparsity of the flow structure of the objects, but also, simultaneously, sparsity of the distance structure of the locations. In several cases this would reduce the number of variables quite further.

One more aspect may make work on the polyhedral combinatorics of the QAP attractive: while investigating properties of the associated polytopes and developing better cutting plane algorithms for the QAP the LP-technology most probably will develop further in parallel. Thus, one might hope that the progress one achieves algorithmically is multiplied by a certain factor that (from the enormous improvements of LP-solvers in the recent years) can be estimated to be not too small.

References

Adams, W. P. and Johnson, T. A. (1994). Improved linear programming-based lower bounds for the quadratic assignment problem. In Pardalos, P. M. and Wolkowicz, H., editors, *Quadratic Assignment and Related Problems*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, pages 43–75.

Barvinok, A. I. (1992). Combinatorial complexity of orbits in representations of the symmetric group. *Advances in Soviet Mathematics*, 9:161–182.

Brüngger, A., Clausen, J., Marzetta, A., and Perregaard, M. (1996). Joining forces in solving large-scale quadratic assignment problems in parallel. Technical Report DIKU TR-23/96, University of Copenhagen, Copenhagen.

Burkard, R. E., Karisch, S. E., and Rendl, F. (1997). QAPLIB — A quadratic assignment problem library. *Journal of Global Optimization*, 10:391–403. `http://serv1.imm.dtu.dk/~sk/qaplib/`.

Chvátal, V. (1983). *Linear Programming*. Freeman.

Clausen, J. and Perregaard, M. (1997). Solving large scale quadratic assignment problems in parallel. *Comput. Optim. Appl.*, 8(2):111–127.

Deza, M. M. and Laurent, M. (1997). *Geometry of Cuts and Metrics*. Springer Verlag.

Edmonds, J. (1965a). Maximum matching and a polyhedron with 0,1-vertices. *Journal of Research of the National Bureau of Standards—B, Mathematics and Mathematical Physics*, 69B:125–130.

Edmonds, J. (1965b). Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17:449–467.

Elf, M. (1999). LP-basierte Schranken für quadratische Zuordnungsprobleme mit dünner Zielfunktion. Master's thesis, Universität zu Köln.

Gilmore, P. C. (1962). Optimal and suboptimal algorithms for the quadratic assignment problem. *SIAM Journal on Applied Mathematics*, 10:305–313.

Grötschel, M., Lovász, L., and Schrijver, A. (1981). The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1:169–197.

Grötschel, M., Lovász, L., and Schrijver, A. (1988). *Geometric Algorithms and Combinatorial Optimization*. Springer-Verlag, Heidelberg.

Grünbaum, B. (1967). *Convex Polytopes*. Interscience, London. Revised edition: V. Klee and P. Kleinschmidt, Graduate Texts in Mathematics, Springer Verlag.

Johnson, T. A. (1992). *New Linear-Programming Based Solution Procedures for the Quadratic Assignment Problem*. PhD thesis, Graduate School of Clemson University.

Jünger, M. and Kaibel, V. (1996). On the SQAP-polytope. Technical Report No. 96.241, Institut für Informatik, Universität zu Köln. Submitted to: SIAM Journal on Optimization.

Jünger, M. and Kaibel, V. (1997a). Box-inequalities for quadratic assignment polytopes. Technical Report 97.285, Angewandte Mathematik und Informatik, Universität zu Köln. Submitted to: Mathematical Programming.

Jünger, M. and Kaibel, V. (1997b). The QAP-polytope and the star-transformation. Technical Report 97.284, Angewandte Mathematik und Informatik, Universität zu Köln. Submitted to: *Discrete Applied Mathematics*.

Jünger, M., Reinelt, G., and Thienel, S. (1994). Provably good solutions for the traveling salesman problem. *ZOR – Mathematical Methods of Operations Research*, 40:183—217.

Kaibel, V. (1997). *Polyhedral Combinatorics of the Quadratic Assignment Problem*. PhD thesis, Universität zu Köln. `www.math.TU-Berlin.de/~kaibel/diss.html`.

Kaibel, V. (1998). Polyhedral combinatorics of QAPs with less objects than locations. In Bixby, R. E., Boyd, E. A., and Rios-Mercado, R. Z., editors, *Proceedings of the 6th International IPCO Conference, Houston, Texas.*, volume 1412 of *Lecture Notes in Computer Science*, pages 409–422. Springer-Verlag.

Karp, R. M. and Papadimitriou, C. H. (1982). On linear characterizations of combinatorial optimization problems. *SIAM Journal on Computing*, 11:620–632.

Khachiyan, L. G. (1979). A polynomial algorithm in linear programming. *Soviet. Math. Dokl.*, 20:191–194.

Klee, V. and Kleinschmidt, P. (1995). Convex polytopes and related complexes. In Graham, R. L., Grötschel, M., and Lovász, L., editors, *Handbook of Combinatorics*, volume 2, chapter 18, pages 875–918. Elsevier Science.

Koopmans, T. C. and Beckmann, M. J. (1957). Assignment problems and the location of economic activities. *Econometrica*, 25:53–76.

Lawler, E. L. (1963). The quadratic assignment problem. *Management Science*, 9:586–599.

Minkowski, H. (1896). *Geometrie der Zahlen.* Teubner Verlag, Leipzig. Reprinted by Chelsea, New York 1953 and Johnson, New York 1963.

Nemhauser, G. L. and Wolsey, L. A. (1988). *Integer and Combinatorial Optimization.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Chichester New York.

Padberg, M. and Rijal, M. P. (1996). *Location, Scheduling, Design and Integer Programming.* Kluwer Academic Publishers.

Padberg, M. W. (1995). *Linear Optimization and Extensions.* Springer-Verlag, Berlin Heidelberg.

Padberg, M. W. and Rao, M. R. (1980). The russian method and integer programming. Technical report, CBA Working Paper, New York University, New York.

Resende, M. G. C., Ramakrishnan, K. G., and Drezner, Z. (1995). Computing lower bounds for the quadratic assignment problem with an interior point solver for linear programming. *Operations Research*, 43:781–791.

Rijal, M. P. (1995). *Scheduling, Design and Assignment Problems with Quadratic Costs.* PhD thesis, New York University.

Schrijver, A. (1986). *Theory of Linear and Integer Programming.* Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons, Chichester New York.

Schrijver, A. (1995). Polyhedral combinatorics. In Graham, R. L., Grötschel, M., and Lovász, L., editors, *Handbook of Combinatorics*, volume 2, chapter 30, pages 1649–1704. Elsevier Science.

Weyl, H. (1935). Elementare Theorie der konvexen Polyeder. *Comm. Math. Helv.*, 7:290–306.

Wolsey, L. A. (1998). *Integer Programming.* John Wiley & Sons, Inc.

Ziegler, G. M. (1995). *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics.* Springer-Verlag, New York. (Revised edition: 1998).

name	initial LP		cutting planes				gap
	bound	qual	bound	qual	iter	time	reduced
chr12a	9552	1.000	9552	1.000	1	16	1.000
chr12b	9742	1.000	9742	1.000	1	16	1.000
chr12c	11156	1.000	11156	1.000	1	21	1.000
had12	1619	0.980	1652	1.000	3	435	1.000
nug12	521	0.901	577	0.997	13	23981	0.971
rou12	222212	0.943	235278	0.999	18	26541	0.981
scr12	29558	0.941	31410	1.000	5	1326	1.000
tai12a	220019	0.980	224416	1.000	3	371	1.000
tai12b	30581825	0.775	39464925	1.000	4	761	1.000
had14	2660	0.976	2724	1.000	4	2781	1.000
chr15a	9371	0.947	9896	1.000	7	25036	1.000
chr15b	7895	0.988	7990	1.000	3	2838	1.000
chr15c	9504	1.000	9504	1.000	1	105	1.000
nug15	1031	0.896	1130	0.982	6	19906	0.827
rou15	322945	0.912	340470	0.961	7	25315	0.561
scr15	48817	0.955	51140	1.000	4	5083	1.000
tai15a	351290	0.905	366466	0.944	7	25449	0.411
tai15b	51528935	0.995	51765268	1.000	7	17909	1.000
esc16b	278	0.952	292	1.000	2	762	1.000
esc16c	118	0.738	160	1.000	4	4929	1.000
esc16h	704	0.707	996	1.000	4	4886	1.000
had16	3549	0.954	3717	0.999	8	23381	0.982
nug16a	1414	0.878	1567	0.973	8	19296	0.781
nug16b	1080	0.871	1209	0.974	5	16512	0.801
nug17	1491	0.861	1644	0.949	4	16007	0.633
tai17a	440095	0.895	454626	0.924	5	25606	0.281
chr18a	10739	0.968	10948	0.986	5	22335	0.580
chr18b	1534	1.000	1534	1.000	1	507	1.000
had18	5072	0.946	5300	0.989	5	23367	0.795
nug18	1650	0.855	1810	0.937	5	19390	0.569
els19	16502857	0.959	17074681	0.992	3	17440	0.806
chr20a	2170	0.990	2173	0.991	2	22488	0.121
chr20b	2287	0.995	2295	0.999	2	13645	0.710
chr20c	14007	0.990	14034	0.992	2	14794	0.196
had20	6559.4	0.948	6732	0.972	2	22783	0.475
lipa20a	3683	1.000	3683	1.000	1	1145	1.000
lipa20b	27076	1.000	27076	1.000	1	935	1.000
nug20	2165	0.842	2314	0.900	3	17845	0.367
rou20	639679	0.882	649748	0.896	3	13143	0.117
scr20	94558	0.859	96562	0.878	3	15122	0.130
tai20a	614850	0.874	625942	0.890	3	34135	0.125
tai20b	84501940	0.690	104534175	0.854	2	10143	0.528

Table 1.1 Results on instances with $m = n$ (dense model).

name	m	opt	init LP	box	iter	time	speed up	ClPer
esc16a	10	68	48	64	3	522	4.87	65
esc16d	14	16	4	16	2	269	2.74	492
esc16e	9	28	14	28	4	588	3.37	66
esc16g	8	26	14	26	3	58	14.62	7
esc16i	9	14	0	14	4	106	28.18	84
esc16j	7	8	2	8	2	25	32.96	14

Table 1.2 Results on the esc16 instances (dense model).

name	m	upper	prev lb	init LP	box	iter	time
esc32a	25	130	36	40	88	3	62988
esc32b	24	168	96	96	100	4	$\star$60000
esc32c	19	642	506	382	506	8	$\star$140000
esc32d	18	200	132	112	152	8	$\star$80000
esc32e	9	2	2	0	2	2	576
esc32f	9	2	2	0	2	2	554
esc32g	7	6	6	0	6	2	277
esc32h	19	438	315	290	352	6	119974

Table 1.3 Results on the esc32 instances (dense model).

name	bound	iter	time
esc32a	92	3	8673
esc32b	96	4	13058
esc32c	394	15	18716
esc32d	120	12	7472
esc32e	2	2	74
esc32f	2	2	82
esc32g	6	4	228
esc32h	280	15	22716

Table 1.4 Results of cutting plane algorithm with “sparse model”.