TWO THEOREMS ON PROJECTIONS OF POLYHEDRA

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Abstract. This note contains two theorems on projections (images under arbitrary linear maps) of polyhedra that are well-known but seem to lack explicit formulations and proofs in the literature. The first result describes the relation between the face lattices of polyhedra and their projections, the second one provides formulae to derive outer descriptions of projected polyhedra from projection cones (for general projections).

Let $P \subseteq \mathbb{R}^n$ be a polyhedron that is the projection $P = \pi(Q)$ of another polyhedron $Q \subseteq \mathbb{R}^d$ under a (in general not one-to-one) linear map $\pi : \mathbb{R}^d \to \mathbb{R}^n$. In this note, we provide proofs of two theorems on such projections that are folklore, but for which we are not aware of a concrete reference in the literature. The first theorem (which, together with its proof, is taken from [2]) describes how the face lattice $\mathcal{L}(P)$ of P arises as a sublattice of the face lattice $\mathcal{L}(Q)$ of Q. The second one provides explicit formulae for deriving an outer description of P from an outer description of Q and knowledge of generators of the *projection cone*. Such formulae seem to be available from the literature only for the special case of orthogonal projections to coordinate subspaces (see, e.g., [1, Sect. 2.4]), where, of course, the general case can in principle be easily obtained from that special case by applying linear transformations. Nevertheless, sometimes it is convenient to have explicit formulae at hand also for the general case, in particular with respect to extended formulations in combinatorial optimization.

If $T \in \mathbb{R}^{n \times d}$ is the matrix such that the projection map $\pi : \mathbb{R}^d \to \mathbb{R}^n$ is given by $\pi(y) = Ty$ for all $y \in \mathbb{R}^d$, then, in the situation described above, a face F of Q is called π-*compatible* if it can be defined by an inequality $\langle T^{\mathrm{t}} a, y\rangle \le \beta$ (valid for Q) for some $a \in \mathbb{R}^n$ (where T^{t} is the transpose of T).

Theorem 1. *For a polyhedron $Q \subseteq \mathbb{R}^d$, a linear projection $\pi : \mathbb{R}^d \to \mathbb{R}^n$, and the polyhedron $P = \pi(Q)$, the map defined via $F \mapsto \pi(F)$ is an isomorphism between the sublattice of $\mathcal{L}(Q)$ formed by the π-compatible faces and $\mathcal{L}(P)$.*

Proof. Let $F \subseteq Q$ be a π-compatible face of Q defined by some inequality $\langle T^{\mathrm{t}} a, y\rangle \le \beta$ (with $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$). For every $y \in Q$, we thus have

$$\langle a, \pi(y)\rangle = \langle a, Ty\rangle = \langle T^{\mathrm{t}} a, y\rangle \le \beta\,,$$

with equality if and only if $y \in F$. Thus, $\langle a, x\rangle \le \beta$ is a valid inequality for $P = \pi(Q)$, defining the face $\pi(F)$ of P. This shows that π indeed maps π-compatible faces of Q to faces of P. Moreover, every face G of P, defined by some inequality $\langle a, x\rangle \le \beta$ (with $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$) is the image of the π-compatible face of Q defined by the inequality $\langle T^{\mathrm{t}} a, x\rangle \le \beta$.

Finally, in order to show that the mapping is one-to-one, let F_1 and F_2 be two π-compatible faces of Q, defined by $\langle T^{\mathrm{t}} a^{(1)}, y\rangle \le \beta_1$ and $\langle T^{\mathrm{t}} a^{(2)}, y\rangle \le \beta_2$, respectively (with $a^{(1)}, a^{(2)} \in \mathbb{R}^n$, $\beta_1, \beta_2 \in \mathbb{R}$), with $\pi(F_1) = \pi(F_2)$. For each $y \in F_1$, we have $\pi(y) \in \pi(F_1) = \pi(F_2)$, thus

$$\beta_2 = \langle a^{(2)}, \pi(y)\rangle = \langle a^{(2)}, Ty\rangle = \langle T^{\mathrm{t}} a^{(2)}, y\rangle\,,$$

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which implies $y \in F_2$. Hence we conclude $F_1 \subseteq F_2$, and similarly, $F_2 \subset F_1$, thus $F_1 = F_2$. $\square$

For a matrix $M \in \mathbb{R}^{r\times s}$ and $f \in \mathbb{R}^r$ we denote the polyhedron defined by $Mx \le f$ by

$$\mathrm{P}^{\le}(M,f) = \{x \in \mathbb{R}^s \,:\, Mx \le f\}\,,$$

and by

$$\ker(M) = \{x \in \mathbb{R}^s \,:\, Mx = \mathbb{O}\}$$

the *kernel* of M. We denote by $M_{i,\star} \in \mathbb{R}^s$ the vector making up the i-th row of M. The (*convex*) *conic hull* of a finite set $Z \subseteq \mathbb{R}^s$ is

$$\operatorname{ccone}(Z) = \Big\{\sum_{z\in Z} \alpha_z z \,:\, \alpha_z \in \mathbb{R}_+ \text{ for all } z \in Z\Big\}$$

(with $\mathbb{R}_+ = \{\alpha \in \mathbb{R} \,:\, \alpha \ge 0\}$). The *image* of the map $\pi : \mathbb{R}^d \to \mathbb{R}^n$ is $\pi(\mathbb{R}^d) = \{\pi(y) \,:\, y \in \mathbb{R}^d\}$.

Theorem 2. *Let $Q = \mathrm{P}^{\le}(D,g) \subseteq \mathbb{R}^d$ be a polyhedron with $D \in \mathbb{R}^{q\times d}$ and $g \in \mathbb{R}^q$, suppose the linear projection $\pi : \mathbb{R}^d \to \mathbb{R}^n$ is defined via $\pi(y) = Ty$ for a matrix $T \in \mathbb{R}^{n\times d}$ and all $y \in \mathbb{R}^d$, and let $\overline{T}$ be any matrix whose rows form a basis of* $\ker(T)$.

If $L \in \mathbb{R}_+^{m\times d}$ is a matrix whose rows generate the projection cone

$$\{\lambda \in \mathbb{R}_+^q \,:\, \lambda^{\mathrm{t}}(D\overline{T}^{\mathrm{t}}) = \mathbb{O}\} = \operatorname{ccone}\{L_{1,\star},\dots,L_{m,\star}\}\,,$$

then every $A \in \mathbb{R}^{m\times n}$ with $AT = LD$ satisfies

$$\pi(Q) = \mathrm{P}^{\le}(A,b) \cap \pi(\mathbb{R}^d)$$

with $b = Lg$.

Proof. The inclusion $\pi(Q) \subseteq \mathrm{P}^{\le}(A,b) \cap \pi(\mathbb{R}^d)$ follows readliy, as, for each $y \in Q = \mathrm{P}^{\le}(D,g)$, we have

$$A \cdot \pi(y) = ATy = LDy \le Lg = b$$

(note that L has nonnegative entries only). In order to establish the reverse inclusion, note that the polyhedron $\pi(Q)$ is contained in the linear subspace $\pi(\mathbb{R}^d)$ of $\mathbb{R}^n$. Thus, one can describe $\pi(Q)$ by linear equations defining $\pi(\mathbb{R}^d)$ and linear inequalities whose left-hand-side coefficient vectors are contained in $\pi(\mathbb{R}^d)$. Therefore, it suffices to show that, for every

$$a \in \pi(\mathbb{R}^d) \text{ and } \beta \in \mathbb{R} \text{ with } \langle a, \pi(y)\rangle \le \beta \text{ for all } y \in Q\,, \tag{1}$$

we have

$$\langle a, x\rangle \le \beta \quad \text{for all } x \in \mathrm{P}^{\le}(A,b) \cap \pi(\mathbb{R}^d) \tag{2}$$

(where, in case of $Q = \varnothing$, it suffices to consider $a = \mathbb{O}$ and $\beta = -1$).

In order to establish this implication, observe that, for a and β as in (1) (with $a = \mathbb{O}$ and $\beta = -1$ in case of $Q = \varnothing$) and for each $y \in Q = \mathrm{P}^{\le}(D,g)$, we have

$$\beta \ge \langle a, \pi(y)\rangle = \langle a, Ty\rangle = \langle T^{\mathrm{t}}a, y\rangle\,,$$

which, by the Farkas-Lemma (see, e.g., [3, Cor. 7.1h] for the case $Q \ne \varnothing$ and [3, Cor. 7.1e] for $Q = \varnothing$), implies that there is some $\lambda \in \mathbb{R}_+^d$ with

$$\lambda^{\mathrm{t}} D = a^{\mathrm{t}} T \quad \text{and} \quad \langle \lambda, g\rangle \le \beta\,.$$

Multiplication of the latter equation by $\overline{T}^{\mathrm{t}}$ yields

$$\lambda^{\mathrm{t}} D\overline{T}^{\mathrm{t}} = a^{\mathrm{t}} T\overline{T}^{\mathrm{t}} = a^{\mathrm{t}}\mathbb{O}_{n\times n} = \mathbb{O}^{\mathrm{t}}\,.$$

Hence, there is some $\mu \in \mathbb{R}^m_+$ with $\mu^{\mathrm{t}} L = \lambda^{\mathrm{t}}$, for which we thus find

$$\mu^{\mathrm{t}} AT = \mu^{\mathrm{t}} LD = \lambda^{\mathrm{t}} D = a^{\mathrm{t}} T\,,$$

which implies $(\mu^{\mathrm{t}} A - a^{\mathrm{t}})T = \mathbb{O}^{\mathrm{t}}$, thus

$$\mu^{\mathrm{t}} A - a^{\mathrm{t}} \in \pi(\mathbb{R}^d)^{\perp}\,.$$

For every $x \in \mathrm{P}^{\leq}(A, b) \cap \pi(\mathbb{R}^d)$, we therefore find

$$\langle a, x\rangle = a^{\mathrm{t}} x + (\mu^{\mathrm{t}} A - a^{\mathrm{t}})x = \mu^{\mathrm{t}} Ax \leq \mu^{\mathrm{t}} b = \mu^{\mathrm{t}} Lg = \lambda^{\mathrm{t}} g \leq \beta\,,$$

which proves (2), and thus, the theorem. □

References

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