Simple 0/1-Polytopes*

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Abstract

For general polytopes, it has turned out that with respect to many questions it suffices to consider only the simple polytopes, i.e., d-dimensional polytopes where every vertex is contained in only d facets. In this paper, we show that the situation is very different within the class of 0/1-polytopes, since every simple 0/1-polytope is the (cartesian) product of some 0/1-simplices (which proves a conjecture of Ziegler), and thus, the restriction to simple 0/1-polytopes leaves only a very small class of objects with a rather trivial structure.

1 Introduction

Polytopes that have only vertices with 0/1-coordinates (*0/1-polytopes*) are important objects that received a lot of attention in fields like combinatorial optimization and integer linear programming. In fact, special 0/1-polytopes that are associated with combinatorial (optimization) problems have extensively been studied in the past. Among the most prominent examples are the *travelling salesman polytopes*, *cut polytopes*, *matching-polytopes*, or *independence polytopes of matroids* (for an overview on the subject of *polyhedral combinatorics* we refer, e.g., to [6]). However, only very few properties of general 0/1-polytopes have been studied yet. Examples for results into this direction are, e.g., the fact that the diameter of a d-dimensional 0/1-polytope may not be larger than d [5], or estimates on the maximal number of facets that a d-dimensional 0/1-polytope can have [4, 3], as well as on the sizes of coefficients of inequalities describing facets of 0/1-polytopes (following from [1]).

With respect to many questions on general polytopes it turned out in the past that it suffices to consider *simple polytopes*, i.e., polytopes, where each vertex is contained in precisely d facets. Examples for such questions are the one for the diameter (*Hirsch conjecture*), the one for the worst-case running time of the simplex algorithm, or the question, how many vertices a polytope with a given number of facets may have (*upper bound theorem*). Furthermore, a simple polytope has the convenient property that its combinatorial structure (i.e. its faces partially ordered by inclusion, called its *face lattice*) is already determined by its *graph* (formed by its vertices and edges). For background material on simple polytopes as well as on polytopes in general, we refer to [7].

In this paper we investigate the simple 0/1-polytopes. It is straightforward to see that every (cartesian) product of 0/1-simplices yields a simple 0/1-polytope. We prove in Sect. 2 that also the converse is true: every simple 0/1-polytope is the product of some simplices. Thus in general the restriction to simple 0/1-polytopes

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will not be sufficient to answer questions on the whole class of 0/1-polytopes, since the simple 0/1-polytopes are a very small and well-structured class of 0/1-polytopes. We list a few more consequences of our result in Sect. 3.

Throughout the paper we will only consider d-dimensional 0/1-polytopes in $\mathbb{R}^d$. This is not much of a restriction, since every 0/1-polytope is affinely isomorphic to a full-dimensional 0/1-polytope by "projecting out" some coordinates. Note moreover that usually we will identify a polytope with the set of its vertices. Finally recall that for a simple polytope P every vertex v together with any set of k neighbours of v in the graph of P span an affine space, whose intersection with P is a k-dimensional face of P. In particular, every vertex has precisely $\dim(P)$ neighbours. We will also use the fact that every face of a simple polytope is simple again.

2 The Main Result

Theorem 1. *A d-dimensional 0/1-polytope $P \subset \mathbb{R}^d$ is simple if and only if it is equal to a (cartesian) product of 0/1-simplices.*

Proof. As mentioned in the introduction, it is clear that every product of 0/1-simplices is a simple 0/1-polytope. To prove the converse let $P \subset \mathbb{R}^d$ be a simple 0/1-polytope. By "flipping coordinates" (which yields an affine isomorphism of the 0/1-hypercube) we can assume that $\mathbf{0}$ (the all-zeroes vector) is a vertex of P.

The crucial feature of simple 0/1-polytopes is the following quite trivial observation.

Lemma 1. *Let $P \subseteq \mathbb{R}^d$ be a simple 0/1-polytope, u a vertex of P and v, w two distinct neighbours of u in the graph of P. Then either v and w are adjacent or $v + w - u$ is a vertex of P, but not both. In the latter case $v + w - u$ is adjacent to v and w.*

Proof. Since P is a simple polytope, the vertices u, v, and w are contained in a 2-dimensional face of P, which is itself a 0/1-polytope. Thus by "projecting out" some coordinates the claim reduces to the case $d = 2$ that is trivial. □

This property of simple 0/1-polytopes has a very restrictive consequence for their graphs.

Lemma 2. *If $P \subseteq \mathbb{R}^d$ is a simple 0/1-polytope and C_1 and C_2 are two cliques in the graph of P such that $C_1 \cup C_2$ is not a clique, then we have*

$$|C_1 \cap C_2| \leq 1 \ .$$

Proof. Assume there exist distinct vertices u and v with $u, v \in C_1 \cap C_2$. Since $C_1 \cup C_2$ is not a clique there exist vertices $r \in C_1$ and $s \in C_2$ which are not adjacent. Because u is adjacent to v, r, s all four points are contained in a 3-dimensional face F of P. Then Lemma 1 (applied to F) implies that the distinct points $q = r + s - v$ and $t = r + s - u$ are vertices of F that are both adjacent to r. Thus, inside the 3-dimensional (simple) face F, r is already adjacent to four distinct vertices, which is a contradiction. □

Let $v \in P$ be a vertex of P. A *star-partition* of P at v is a collection

$$\{v\} \cup S_i \subset P \qquad (i = 1, \dots, r)$$

of maximal cliques in the graph of P such that $S_1 \cup \dots \cup S_r$ is a partition of the neighbours of v ($S_i \cap S_j = \emptyset$ for $i \neq j$). From Lemma 2 it follows that for $i \neq j$ any vertices $u \in S_i$ and $x \in S_j$ cannot be adjacent. Thus P has a unique star-partition at every vertex v.

Let $\{\mathbf{0}\} \cup S_i$ $(i = 1, \ldots, r)$ be the star-partition of P at $\mathbf{0}$. Suppose for $i \neq j$ the vertices $u \in S_i$ and $x \in S_j$ both are equal to 1 in a common coordinate. Hence, $u + x \notin P$, and therefore, by Lemma 1, u and x must be adjacent, which, however, is not true due to $i \neq j$. Thus, there is a partition

$$I_1 \uplus \cdots \uplus I_r = \{1, \ldots, d\}$$

$(I_i \cap I_j = \emptyset$ for $i \neq j)$ such that

$$\mathrm{supp}(u) \subseteq I_i$$

for each $i \in \{1, \ldots, r\}$ and $u \in S_i$ (where $\mathrm{supp}(u)$ denotes the set of coordinates where u is non-zero). Actually, since P has dimension d, we even have

$$\biguplus_{u \in S_i} \mathrm{supp}(u) = I_i$$

for every $i \in \{1, \ldots, r\}$ and thus $I_1, \ldots, I_r$ are also determined uniquely.

For a subset $J \subseteq \{1, \ldots, d\}$ of coordinates we denote by $\mathbb{R}^J$ the $|J|$-dimensional real vector-space whose vectors are indexed by the elements of J. In particular, we do not identify $\mathbb{R}^J$ and $\mathbb{R}^{|J|}$. The map $\pi_J : \mathbb{R}^d \longrightarrow \mathbb{R}^J$ is the orthogonal projection that eliminates all components of a vector that do not belong to J. We will be especially concerned with the sets

$$\Delta_i := \pi_{I_i}(\{\mathbf{0}\} \cup S_i) \qquad (i = 1, \ldots, r) \ .$$

Because in a simple polytope a vertex and a subset of its neighbours are always affinely independent, all Δ_i are (vertex sets of) 0/1-simplices.

For $v \in \mathbb{R}^d$ and $u \in \mathbb{R}^J$ we denote by

$$(v \,|\, u) \in \mathbb{R}^d$$

the vector with

$$(v \,|\, u)_a := \begin{cases} u_a & \text{if } a \in J \\ v_a & \text{otherwise} \end{cases} \ .$$

Using these notations, the star-partition of P at $\mathbf{0}$ is given by

$$\{(\mathbf{0} \,|\, u) : u \in \Delta_i\} \qquad (i = 1, \ldots, r) \ . \tag{1}$$

The proof will be almost finished once we have shown that the structure of the neighbours of a vertex v of P given by the star-partition at v carries over to the structure of every neighbour of v. This is stated in the following lemma (see Figure 1).

Lemma 3. *Suppose that $v \in P$ is a vertex of P with star-partition*

$$\{(v \,|\, u) : u \in \Delta_i\} \qquad (i = 1, \ldots, r) \ .$$

Then every neighbour $w \in P$ of v has star-partition

$$\{(w \,|\, x) : x \in \Delta_j\} \qquad (j = 1, \ldots, r) \ . \tag{2}$$

Proof. Let $w = (v \,|\, u)$ $(u \in \Delta_i,\ i \in \{1, \ldots, r\})$ be a neighbour of v. Clearly, for all $x \in \Delta_i \setminus \{u\}$ the vertex $(w \,|\, x) = (v \,|\, x)$ is a neighbour of w. Let $x \in \Delta_j$ $(j \neq i)$ with $(v \,|\, x) \neq v$. Since $(v \,|\, u)$ and $(v \,|\, x)$ are not adjacent (but they are both neighbours of v),

$$(v \,|\, u) + (v \,|\, x) - v = (w \,|\, x)$$

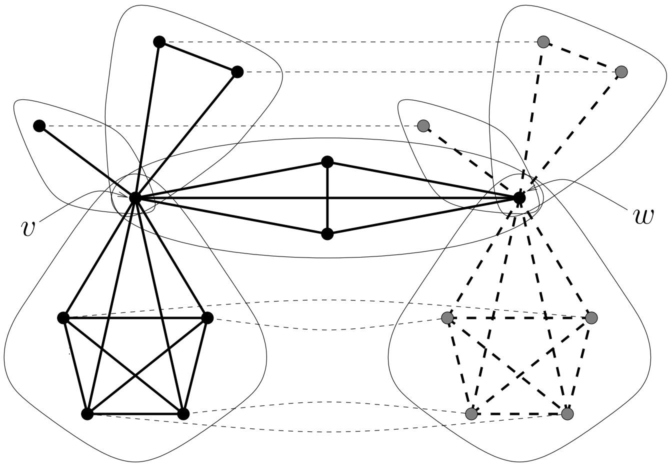

Figure 1: The star-partition at v carries over to the star-partition at the neighbour w of v (Lemma 3).

must (by Lemma 1) be a vertex of P that is adjacent to $(v \,|\, u) = w$ (and also to $(v \,|\, x)$). Hence we have already identified all d neighbours of w. Thus the vertices in (2) are indeed precisely the neighbours of w.

It remains to show that two neighbours of v are adjacent if and only if the corresponding neighbours of w are adjacent. Let $(w \,|\, x)$ and $(w \,|\, x')$ be two distinct neighbours of w with $x \in \Delta_k$ and $x' \in \Delta_l$. Let us first consider the case that $k \neq i$ and $l \neq i$. Since we know already that (2) gives the neighbours of neighbours of v, $(v \,|\, x)$ and $(w \,|\, x)$ as well as $(v \,|\, x')$ and $(w \,|\, x')$ then are adjacent. Furthermore, we have

$$(v \,|\, x) + (w \,|\, x') = (v \,|\, x) + ((v \,|\, u) \,|\, x') = (v \,|\, x') + ((v \,|\, u) \,|\, x) = (v \,|\, x') + (w \,|\, x) \ .$$

Thus by Lemma 1 $(w \,|\, x)$ and $(w \,|\, x')$ are adjacent if and only if $(v \,|\, x)$ and $(v \,|\, x')$ are adjacent. If $k = l = i$ then $(w \,|\, x)$ and $(w \,|\, x')$ clearly are adjacent. Thus the final case to be considered is w.l.o.g. $k = i$ and $l \neq i$. But then $(w \,|\, x')$ is not among the neighbours of the neighbour $(v \,|\, x) = (w \,|\, x)$ of v. Hence (2) is indeed the star-partition of w. □

Now we can finish the proof of Theorem 1. Since (1) gives the star-partition of the vertex $\mathbf{0}$ of P, by inductive applications of Lemma 3 all vertices of $\Delta_1 \times \cdots \times \Delta_r$ turn out to be vertices of P. On the other hand, again by Lemma 3, each of these vertices of P has already all its neighbours in $\Delta_1 \times \cdots \times \Delta_r$, which finally yields

$$P = \Delta_1 \times \cdots \times \Delta_r \ ,$$

since the graph of P is connected. □

3 Conclusions and Remarks

Since for two polytopes P and Q with f_P and f_Q facets, respectively, the product $P \times Q$ has $f_P + f_Q$ facets, the following is a trivial consequence of Theorem 1.

Corollary 1. *A simple 0/1-polytope of dimension d has at most $2d$ facets. It has $2d$ facets if and only if it is the d-dimensional hypercube.*

Besides simple polytopes, also *simplicial polytopes* (i.e., polytopes, where every proper face is a simplex) are of special interest. Two polytopes, whose face lattices are obtained from each other by reversing the order relation, are called *polar* to each other. It is well-known, that a polytope is simple if and only if its polar is simplicial. Unfortunately, in general the polar of a 0/1-polytope P cannot be represented as a 0/1-polytope. That means that in general there is no 0/1-polytope that is *combinatorially isomorphic* (i.e., the face lattices are isomorphic) to the polar of P. However, for simple 0/1-polytopes the following holds.

Corollary 2. *The polar of a simple 0/1-polytope is (combinatorially) isomorphic to a (simplicial) 0/1-polytope.*

Proof. Let $S_1, S_2 \subset \mathbb{R}^d$ be two affine subspaces that span the whole space $\mathbb{R}^d$ with $S_1 \cap S_2 = \{b\}$. Let $P \subset S_1$ and $Q \subset S_2$ be two polytopes that both contain b as an interior point. Then we define the *free sum* $P \oplus Q$ of P and Q to be the polytope

$$P \oplus Q = \operatorname{conv}(P \cup Q) \ .$$

It is well-known (see, e.g., [4]) that if P and Q as above are polar to P' and Q', respectively, then $P \oplus Q$ is polar to $P' \times Q'$. Thus in order to prove the corollary, we shall show how to realize free sums of 0/1-simplices (that are, of course, polar to itself) as 0/1-polytopes. Since we are interested only in combinatorial isomorphisms, we are free to choose the concrete 0/1-coordinates of the simplices as long as their dimensions are preserved.

Unfortunately, we cannot apply the free sum construction of [4] directly because there is no 2-dimensional 0/1-simplex which contains the point $(1/2, 1/2)$ in its interior. Instead we use the following slight modification of the construction of [4]. Suppose we have to realize a free sum of 0/1-simplices of dimensions $d_1, \ldots, d_r$, where we may assume $1 \leq d_1 \leq \cdots \leq d_r$. We will inductively realize for $j = 1, \ldots, r$ the free sum of the first j simplices as a $D_j = d_1 + \cdots + d_j$-dimensional 0/1-polytope P_j. For technical reasons, we will do this in such a way that the point $\mathbf{0}$ is one of the vertices of P_j and the point $(1/(d_j+1), \ldots, 1/(d_j+1))$ is located in the interior of P_j. Notice that this is no restriction for the first simplex.

For $j \in \{1, \ldots, r-1\}$ the simplex Δ_{j+1} is realized as the convex hull of the first d_{j+1} unit vectors and the vector which is 0 in the first d_{j+1} coordinates and 1 in the remaining ones. This means that Δ_{j+1} is embedded in the affine space

$$S_1 = \{x \in \mathbb{R}^{D_{j+1}} : x_1 + \cdots + x_{d_{j+1}+1} = 1 \text{ and } x_{d_{j+1}+1} = \cdots = x_{D_{j+1}}\} \ .$$

The polytope P_j is embedded as P'_j in the affine space

$$S_2 = \{x \in \mathbb{R}^{D_{j+1}} : x_1 = \cdots = x_{d_{j+1}+1}\} \ ,$$

which is possible since S_2 contains a D_j-dimensional 0/1-cube. The intersection of the two affine spaces is the point $(1/(d_{j+1}+1), \ldots, 1/(d_{j+1}+1))$ which is contained in the interior of both the polytopes Δ_{j+1} and P'_j, in the latter case because P'_j contains the point $(1/(d_j+1), \ldots, 1/(d_j+1))$ in its interior as well as the vertex $\mathbf{0}$ and we have $d_{j+1} \geq d_j$. Clearly, the new polytope P_{j+1} again contains both the vertex $\mathbf{0}$ and the interior point $(1/(d_{j+1}+1), \ldots, 1/(d_{j+1}+1))$. □

Fortunately it is not true, that every simplicial 0/1-polytope is the polar of a simple 0/1-polytope. For instance, there is a 4-dimensional simplicial 0/1-polytope with 7 vertices and 13 facets, however, 13 is prime and the 4-dimensional simplex does not have 13 vertices (and, of course, if P and Q are polytopes with v_P and v_Q vertices, respectively, the product $P \times Q$ has $v_P \cdot v_Q$ vertices). Therefore, it is possible that the class of simplicial 0/1-polytopes has a less trivial structure, and maybe one can learn much more about general 0/1-polytopes from the simplicial ones than from the simple ones. Hence the problem of classifying the simplicial 0/1-polytopes remains to be an interesting open question.

We conclude with one more consequence of Theorem 1. Mihail and Vazirani conjectured that graphs of 0/1-polytopes have *cutset expansion* one, i.e., the cardinality of every cut is not smaller than the cardinality of the smaller of the two vertex sets separated by the cut. See [2] for a citation of this conjecture. One can easily prove this conjecture for products of simplices and hence for simple 0/1-polytopes.

Corollary 3. *The graph of a simple 0/1-polytope P has cutset expansion one.*

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