8

Automorphism Groups of Cyclic Polytopes

(Volker Kaibel[1] and Arnold Waßmer[2])

It is probably well-known to most polytope theorists that the combinatorial automorphism group of a cyclic d-polytope with n vertices is isomorphic to $\mathbb{D}_n$ (the dihedral group of order n) for even d and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for odd d — for $n \geq d+3$. However, it seems that this is not accessible at any prominent place in the literature. Our detailed elaboration here may not only serve as an example for the determination of combinatorial automorphism groups of polytopes, but it also reveals the variational structures for $n = d+2$. Moreover, we show that each cyclic polytope can be realized in Euclidean space such that all its combinatorial automorphisms are induced by self-congruences.

8.1 Isomorphisms of Polytopes

A *combinatorial isomorphism* between two (convex) polytopes P and Q is an isomorphism between their *face lattices* (i.e., the sets of faces, partially ordered by inclusion). Two polytopes are *combinatorially equivalent* ($P \cong Q$) if there is a combinatorial isomorphism between them. Since every face of a polytope is the convex hull of its vertices (the face lattice is atomic), one may identify combinatorial isomorphisms of P and Q with bijections between the vertex sets of P and Q that preserve vertex sets of faces. Furthermore, every face of a convex polytope is the intersection of all facets it is contained in (the face lattice is coatomic). Thus, for combinatorial equivalence of polytopes, it suffices to consider the vertex sets of facets only. In other words, the combinatorial type of a polytope only depends on its *vertex-facet incidences.*

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Particularly interesting are the *combinatorial automorphisms* of a polytope P (the combinatorial isomorphisms between P and itself), i.e., the group $\mathrm{AUT}(P)$ of permutations of the vertex set of P that map vertex sets of facets to vertex sets of facets.

8.2 Cyclic Polytopes

A d-dimensional polytope is *cyclic* if it is combinatorially equivalent to the convex hull of n distinct points on the moment curve

$$\gamma_{\mathrm{M}} \ : \ \mathbb{R} \ni t \mapsto (t, t^2, \dots, t^d) \in \mathbb{R}^d \ .$$

Here, the actual choice of the n points (which turn out to be the vertices of the convex hull) is irrelevant for the combinatorial type (see Theorem 8.1), which usually is denoted by $C_d(n)$. Every d-dimensional cyclic polytope with n vertices is called a *realization* of $C_d(n)$. For example, in even dimensions d, the convex hull of n distinct points on the Carathéodory curve

$$\gamma_{\mathrm{C}} \ : \ \mathbb{R} \ni t \mapsto \big(\sin(t), \cos(t), \sin(2t), \cos(2t), \dots, \sin(\tfrac{d}{2}t), \cos(\tfrac{d}{2}t)\big) \in \mathbb{R}^d$$

is a realization of $C_d(n)$ [5, Ex. 2.21].

Cyclic polytopes are important, since they are the ones with the most easily accessible combinatorial structure among the *neighborly* d-dimensional polytopes (i.e., every subset of the vertices with cardinality at most $\frac{d}{2}$ is the vertex set of a face). Among the d-dimensional polytopes with n vertices, the neighborly polytopes are the ones with most i-dimensional faces for each i (by McMullen's upper bound theorem, see [5, Thm. 8.23]).

In order to describe the combinatorial structure of $C_d(n)$, let $[n] := \{1, \dots, n\}$ be the set of vertices, numbered according to the order induced by the moment curve.

Let $F \subset [n]$ be a subset of vertices of cardinality $|F| = d$. We represent F by its characteristic vector, which we imagine as a $(1 \times n)$-table (its *star table*) having a '$\star$' or an empty entry at position i, depending on whether $i \in F$ or not. In such a star table, every maximal group of consecutive stars is called a *block*. A block containing one vertex from $\{1, n\}$ is a *border block*; the other ones are *inner blocks*.

For example the set $\{1, 2, 3, 5, 6, 8, 9, 10, 11, 15\} \subset [15]$ in the star table looks like

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\star$	$\star$	$\star$		$\star$	$\star$		$\star$	$\star$	$\star$	$\star$				$\star$

with border blocks $\{1, 2, 3\}$ and $\{15\}$ and inner blocks $\{5, 6\}$ and $\{8, 9, 10, 11\}$.

The following description of the combinatorial structures of the cyclic polytopes is due to Gale [1] (see [5, Thm. 0.7]).

Theorem 8.1 (Gale's evenness criterion)
A set $F \subset [n]$ corresponds to the vertex set of a facet of $C_d(n)$ if and only if $|F| = d$ and in the star table of F, all inner blocks have even size.

Theorem 8.1 in particular shows that cyclic polytopes are simplicial.

For the cyclic polytopes shown in Figure 8.2, Gale's evenness criterion can easily be verified. One may already see the symmetries of the cyclic polytopes.

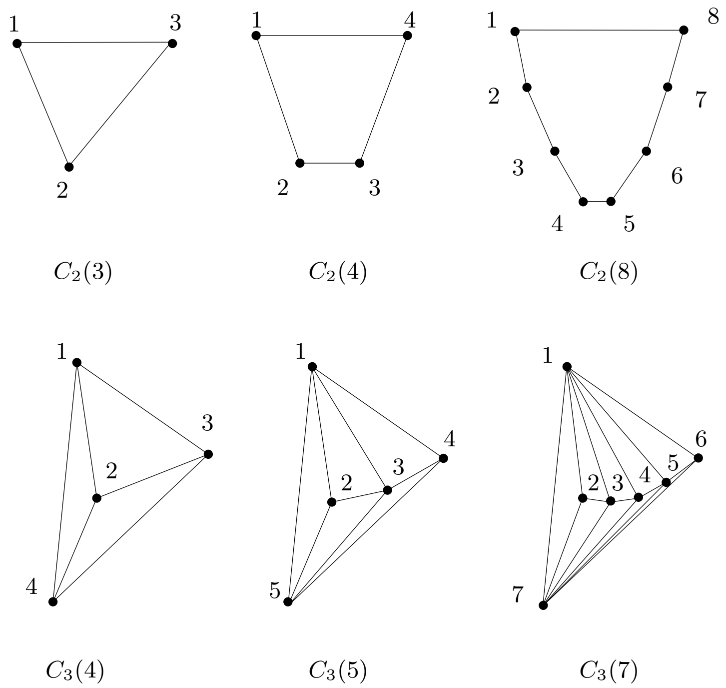


Fig. 8.1. Small examples of cyclic polytopes.

For even dimensions d, we glue together the two ends of the star table (as well as the border blocks) to form a *star circle* as shown in figure 8.2. The border blocks of facets have the same parity for even d. Therefore, Gale's evenness criterion can be stated more symmetrically in this case.

Theorem 8.2 (Gale's evenness criterion for even dimensions)
For even d, a set $F \subset [n]$ corresponds to the vertex set of a facet of $C_d(n)$ if and only if $|F| = d$ and the star circle of F has only blocks of even size.

This cyclic symmetry in even dimensions fits together with the realization of $C_d(n)$ via the Carathéodory curve γ_C, which is a closed curve, and thus, induces a cyclic rather than a linear order on the vertices.

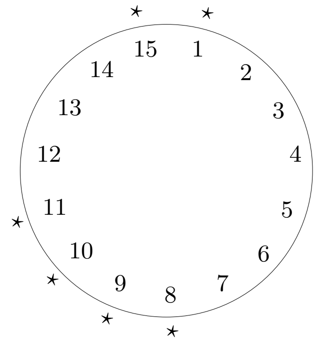


Fig. 8.2. A facet in the star circle of the even dimensional cyclic polytope $C_6(15)$.

Theorem 8.3 (Automorphism groups of cyclic polytopes)
The combinatorial automorphism group of a d-dimensional cyclic polytope with n vertices is, depending on n and on the parity of d, isomorphic to one of the following groups:

	$n = d+1$	$n = d+2$	$n \geq d+3$
d even	$\mathbb{S}_n$	$\mathbb{S}_{\frac{n}{2}}$ wr $\mathbb{Z}_2$	$\mathbb{D}_n$
d odd	$\mathbb{S}_n$	$\mathbb{S}_{\lceil \frac{n}{2} \rceil} \times \mathbb{S}_{\lfloor \frac{n}{2} \rfloor}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

The corresponding group action is vertex transitive if and only if d is even or $n = d+1$.

In the above table, we denote by $\mathbb{S}_k$ the symmetric group on k elements, i.e., the group of all permutations of the set $[k]$. The group $\mathbb{D}_n$ is the dihedral group, the symmetry group of the regular n-gon. The *wreath product* $\mathbb{S}_{\frac{n}{2}}$ wr $\mathbb{Z}_2$ is a certain semi-direct product of $\mathbb{S}_{\frac{n}{2}} \times \mathbb{S}_{\frac{n}{2}}$ and $\mathbb{Z}_2$ (see below).

The following theorem states that cyclic polytopes have symmetric realizations (where a *congruence* between two polytopes is a bijection induced by an orthogonal affine transformation).

Theorem 8.4 (Self-congruences of cyclic polytopes)
All cyclic polytopes $C_d(n)$ can be realized in Euclidean space such that all their combinatorial automorphisms are induced by self-congruences.

In the following sections, we will prove Theorems 8.3 and 8.4 separately for the cases $n = d+2$ and $n \geq d+3$. The case $n = d+1$ is trivial since the cyclic polytope $C_d(d+1)$ is the simplex. Its realization for Theorem 8.4 is any regular simplex.

8.3 The case $n = d + 2$

The central observation (see [5, Ex. 0.6]) for this case is that $C_d(d+2)$ can be realized as a *free sum* of two simplices, where the free sum of two polytopes P and Q lying in two orthogonal subspaces and having a unique common relative interior point is $P \oplus Q := \operatorname{conv}(P \cup Q)$. It is well-known that $P \oplus Q$ and $P \times Q$ are dual to each other. Thus, the set of vertices of $P \oplus Q$ can be identified with the union $\operatorname{vert}(P) \cup \operatorname{vert}(Q)$ of the sets of vertices of P and Q, where $F \subset \operatorname{vert}(P) \cup \operatorname{vert}(Q)$ corresponds to a facet of $P \oplus Q$ if and only if $F \cap \operatorname{vert}(P)$ is the vertex set of a facet of P and $F \cap \operatorname{vert}(Q)$ is the vertex set of a facet of Q. See [2] and [3] for discussions of the free sum construction.

Proposition 8.5 *The free sum of two simplices of dimensions $\lceil \frac{d}{2} \rceil$ and $\lfloor \frac{d}{2} \rfloor$, respectively, is a d-dimensional cyclic polytope with $d + 2$ vertices.*

Proof. Each facet of $C_d(d+2)$ contains d vertices. By Gale's evenness criterion, the two missing vertices define an inner block of even size in the corresponding star table. Hence they have different parity. The facets of $C_d(d+2)$ therefore are

$$\left\{ ([d+2]_{\text{odd}} \setminus \{o\}) \cup ([d+2]_{\text{even}} \setminus \{e\}) \ : \ o \in [d+2]_{\text{odd}},\ e \in [d+2]_{\text{even}} \right\}$$

with $[n]_{\text{odd}} := [n] \cap (2\mathbb{Z}+1)$ and $[n]_{\text{even}} := [n] \cap (2\mathbb{Z})$. In fact, this is precisely the vertex-facet incidence structure of the free sum of two simplices of dimensions $\lceil \frac{d}{2} \rceil$ and $\lfloor \frac{d}{2} \rfloor$, respectively, one having its vertices labeled by $[d+2]_{\text{odd}}$, and the other one by $[d+2]_{\text{even}}$. This proves Proposition 8.5.

For example, the cyclic polytope $C_2(4)$ is a 4-gon. It is the free sum of two orthogonal line segments. A 3-dimensional cyclic polytope of the type $C_3(5)$ is depicted in figure 8.3. One can see that it is the free sum of the 1-simplex $\operatorname{conv}\{2, 4\}$ and the 2-simplex $\operatorname{conv}\{1, 3, 5\}$.

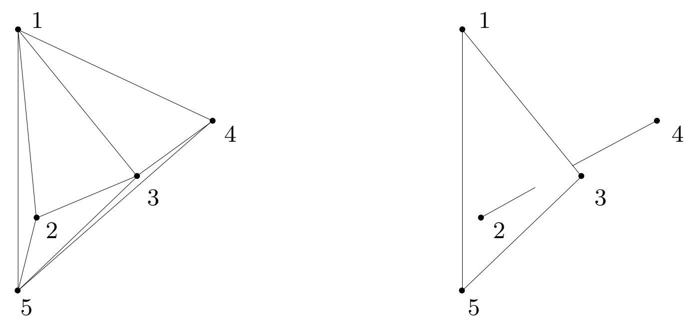


Fig. 8.3. The cyclic polytope $C_3(5)$ is the free sum of two simplices.

Combinatorial Automorphisms

It follows from Proposition 8.5 (and its proof) that the combinatorial automorphisms of $C_d(d+2)$ are the permutations of $[d+2] = [d+2]_{\text{odd}} \cup [d+2]_{\text{even}}$ that either respect $[d+2]_{\text{odd}}$ and $[d+2]_{\text{even}}$ or map $[d+2]_{\text{odd}}$ to $[d+2]_{\text{even}}$ (and vice versa), where the latter situation is possible only for even d. In particular, the combinatorial automorphism group acts vertex transitively if and only if d is even.

Let $\mathbb{S}^{\text{odd}}_{d+2}$ and $\mathbb{S}^{\text{even}}_{d+2}$ be the groups of all permutations of the sets $[d+2]_{\text{odd}}$ and $[d+2]_{\text{even}}$, respectively, viewed as subgroups of the group $\mathbb{S}_{d+2}$ of permutations of $[d+2]$. Note that $\sigma_{\text{odd}} \circ \sigma_{\text{even}} = \sigma_{\text{even}} \circ \sigma_{\text{odd}}$ for every $\sigma_{\text{odd}} \in \mathbb{S}^{\text{odd}}_{d+2}$, $\sigma_{\text{even}} \in \mathbb{S}^{\text{even}}_{d+2}$. For odd d, we clearly have

$$\text{AUT}(C_d(d+2)) \;=\; \mathbb{S}^{\text{odd}}_{d+2} \times \mathbb{S}^{\text{even}}_{d+2} \;\cong\; \mathbb{S}_{\lceil \frac{d}{2} \rceil} \times \mathbb{S}_{\lfloor \frac{d}{2} \rfloor} \;.$$

For even d, the group structure of the combinatorial automorphisms is a bit more complicated. The permutation $t := (1\,2)(3\,4)\ldots(d+1\,,d+2)$ generates a group $T := \langle t \rangle = \{\text{id}, t\} \cong \mathbb{Z}_2$. For $\sigma_{\text{even}}, \sigma'_{\text{even}} \in \mathbb{S}^{\text{even}}_{d+2}$ the relation

$$t \circ \sigma'_{\text{even}} \circ t \circ \sigma_{\text{even}} = \sigma_{\text{even}} \circ t \circ \sigma'_{\text{even}} \circ t$$

holds (similar relations are valid for the odd parts, of course). Finally, let us define the group homomorphism

$$\begin{array}{rcl} \varPhi :\; T & \longrightarrow & \text{AUT}(\mathbb{S}^{\text{odd}}_{d+2} \times \mathbb{S}^{\text{even}}_{d+2}) \\ \tau & \longmapsto & \varPhi_\tau \;, \end{array}$$

where $\varPhi_{\text{id}} := \text{id}$, and $\varPhi_t$ is defined via $(\sigma_{\text{odd}}, \sigma_{\text{even}}) \mapsto (t \circ \sigma_{\text{even}} \circ t\;,\; t \circ \sigma_{\text{odd}} \circ t)$.

We can write each element of $\text{AUT}(C_d(d+2))$ uniquely as $\sigma_{\text{odd}} \circ \sigma_{\text{even}} \circ \tau$ with $\sigma_{\text{odd}} \in \mathbb{S}^{\text{odd}}_{d+2}$, $\sigma_{\text{even}} \in \mathbb{S}^{\text{even}}_{d+2}$, and $\tau \in T$. This defines a bijection β between the *sets* $\text{AUT}(C_d(d+2))$ and $(\mathbb{S}^{\text{odd}}_{d+2} \times \mathbb{S}^{\text{even}}_{d+2}) \times T$. The group structure induced by this bijection on the set $(\mathbb{S}^{\text{odd}}_{d+2} \times \mathbb{S}^{\text{even}}_{d+2}) \times T$ is given by

$$\begin{array}{l} ((\sigma_{\text{odd}}, \sigma_{\text{even}})\;,\; \tau) \circ ((\sigma'_{\text{odd}}, \sigma'_{\text{even}})\;,\; \tau') \\ \qquad = \; \beta(\sigma_{\text{odd}} \circ \sigma_{\text{even}} \circ \tau) \circ \beta(\sigma'_{\text{odd}} \circ \sigma'_{\text{even}} \circ \tau') \\ \qquad = \; \beta(\sigma_{\text{odd}} \circ \sigma_{\text{even}} \circ \tau \circ \sigma'_{\text{odd}} \circ \sigma'_{\text{even}} \circ \tau') \\ \qquad = \; ((\sigma_{\text{odd}}, \sigma_{\text{even}}) \circ \varPhi_\tau(\sigma'_{\text{odd}}, \sigma'_{\text{even}})\;,\; \tau \circ \tau') \;. \end{array}$$

This group structure is known as the *semi-direct product* $(\mathbb{S}^{\text{odd}}_{d+2} \times \mathbb{S}^{\text{even}}_{d+2}) \rtimes_{\varPhi} T$ of $(\mathbb{S}^{\text{odd}}_{d+2} \times \mathbb{S}^{\text{even}}_{d+2})$ and T with respect to $\varPhi$.

With the group homomorphism $\tilde{\varPhi} \;:\; \mathbb{Z}_2 \longrightarrow \mathbb{S}_{\frac{d+2}{2}} \times \mathbb{S}_{\frac{d+2}{2}}$ corresponding to $\varPhi$, we thus have

$$\text{AUT}(C_d(n)) \;\cong\; (\mathbb{S}_{\frac{d+2}{2}} \times \mathbb{S}_{\frac{d+2}{2}}) \rtimes_{\tilde{\varPhi}} \mathbb{Z}_2 \;=:\; \mathbb{S}_{\frac{d+2}{2}} \text{ wr } \mathbb{Z}_2 \;,$$

where the latter group is called the *wreath product* of $\mathbb{S}_{\frac{d+2}{2}}$ and $\mathbb{Z}_2$.

Already for $d = 2$, the semi-direct product $\mathrm{AUT}(C_2(4)) \cong (\mathbb{S}_2 \times \mathbb{S}_2) \rtimes_{\tilde{\Phi}} \mathbb{Z}_2$ is not isomorphic to the direct product $(\mathbb{S}_2 \times \mathbb{S}_2) \times \mathbb{Z}_2$, as all elements in the latter group have order at most two, while $\mathrm{AUT}(C_2(4))$ has an element of order four (the rotation by $\frac{\pi}{2}$, if $C_2(4)$ is realized as a square).

Geometric Realizations

Due to Proposition 8.5, we may realize the cyclic polytope $C_d(d+2)$ as a free sum of two appropriate simplices $\Delta_{\lceil \frac{d}{2} \rceil}, \Delta_{\lfloor \frac{d}{2} \rfloor} \subset \mathbb{R}^d$. We choose $\Delta_{\lceil \frac{d}{2} \rceil}$ and $\Delta_{\lfloor \frac{d}{2} \rfloor}$ to be regular simplices with all edges having length one, located in two orthogonal linear subspaces of $\mathbb{R}^d$, such that both have the origin as their barycenter. Then all combinatorial automorphisms of $\Delta_{\lceil \frac{d}{2} \rceil} \oplus \Delta_{\lfloor \frac{d}{2} \rfloor}$ are induced by self-congruences.

8.4 The case $n \geq d+3$

Throughout this section, let $n \geq d+3$.

Combinatorial Automorphisms

Shemer [4] calls the pairs

$$\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\} \qquad \text{(for even } d\text{)}$$

$$\{2,3\}, \ldots, \{n-2,n-1\} \qquad \text{(for odd } d\text{)}$$

the *universal edges* of $C_d(n)$ (it is not hard to see that these pairs correspond to one-dimensional faces of $C_d(n)$, i.e., they form edges in the graph of $C_d(n)$). A vertex w is a *strong neighbor* of another vertex v of $C_d(n)$ if $\{v,w\}$ is a universal edge. For odd d, the vertices 1 and n are the *border vertices*, and 2, $\ldots$, $n-1$ are the *inner vertices.*

Lemma 8.6 *For odd d, each facet of $C_d(n)$ contains at least one of the two border vertices.*

Proof. A facet of $C_d(n)$ contains an odd number of vertices, namely d of them. According to Gale's evenness criterion the inner blocks of its star table have even size. Hence there is at least one odd border block. This proves the lemma.

For odd dimensions the following lemma is complementary to this.

Lemma 8.7 (Two forbidden vertices) *Let x and y be two vertices of $C_d(n)$, at most one of which is a border vertex in case that d is odd. Then there is a facet of $C_d(n)$ which neither contains x nor y.*

Proof. If d is even, then x and y split up the circle of Gale's evenness criterion into two intervals I_1 and I_2 of length l_1 and l_2, respectively, with $l_1+l_2 = n-2$. For each $i = 1, 2$, the interval I_i contains an even interval $\tilde{I}_i$ of size $\tilde{l}_i := l_i - (l_i \mod 2)$. Due to $\tilde{l}_1 + \tilde{l}_2 = n - 2 - \big((l_1 \mod 2) + (l_2 \mod 2)\big) \geq d$ (where the inequality is clear for $n > d + 3$ and follows from the fact that d is even for $n = d + 3$), the desired facet exists.

If d is odd, then we may assume that the border vertex n is neither x nor y. Hence it suffices to apply the lemma to $C_{d-1}(n-1)$ (with even $d-1$), x, and y. This proves Lemma 8.7.

Lemma 8.6 and Lemma 8.7 characterize the two border vertices of an odd dimensional cyclic polytope as the only pair of vertices that intersects each facet. This implies the following

Corollary 8.8 *For odd d, each combinatorial automorphism of $C_d(n)$ fixes both the set of border vertices and the set of inner vertices.*

Let us consider the star table (or star circle) of a facet F of $C_d(n)$ with $v, w \in F$ being strong neighbors of each other. If we delete v and w from this star table together with their columns, then we obtain (after adjusting indices) the star table of a facet of $C_{d-2}(n-2)$. Conversely, if we insert two neighboring columns with stars into the star table of a facet of $C_{d-2}(n-2)$, then we get a facet of $C_d(n)$.

With this observation we can extend Lemma 8.7 to the following.

Lemma 8.9 (Two forbidden vertices and two neighbors) *Let $C_d(n)$ be a cyclic polytope with $n \geq d + 3$ vertices. Let x and y be two vertices of it, at most one of which is a border vertex (if d is odd). Let v and w be a pair of strong neighbors with $\{v, w\} \cap \{x, y\} = \varnothing$. Then there is a facet of $C_d(n)$ that contains both the vertices v and w but neither x nor y.*

Now we have all tools in place to prove the following proposition.

Proposition 8.10 *For $n \geq d + 3$, every combinatorial automorphism of $C_d(n)$ maps pairs of strong neighbors to pairs of strong neighbors.*

Proof. Let us first consider the case of even d. Suppose a and b form a pair of strong neighbors mapped by some combinatorial automorphism of $C_d(n)$ to vertices a' and b' that do not form a pair of strong neighbors. Thus, a' has two strong neighbors x' and y', both different from b'. According to Lemma 8.9, we can find a facet of $C_d(n)$ that does not contain the preimages x and y of x' and y', but that does contain the pair a and b. The image of this facet contains an odd block $\{a'\}$, contradicting the fact that it is a facet.

If d is odd, suppose again that the vertices a and b form a pair of strong neighbors, but, under some combinatorial automorphism, their images a' and b' do not. Since a and b are strong neighbors to each other and only inner vertices can have strong neighbors, a and b are inner vertices. The set of inner vertices is fixed under every combinatorial automorphism. Hence a' and b' are inner vertices, too. As in the case before, the vertices $x' := a' - 1 \in [n]$ and $y' := a' + 1 \in [n]$ both are disjoint from b'. Since a' is an inner vertex, one of x' and y' is an inner vertex, too. The same holds for one of their preimages x and y. Thus, also in this case, we can derive a contradiction from Lemma 8.9, and Proposition 8.10 is proven.

Proposition 8.10 implies severe restrictions to the combinatorial automorphism group of the cyclic polytope $C_d(n)$ if $n \geq d + 3$. If the dimension d is even, then the pairs of strong neighbors form a cycle of length n in the graph of $C_d(n)$ that must be mapped to itself by the induced graph automorphism. Hence the group $\mathrm{AUT}(C_d(n))$ can be at most the dihedral group $\mathbb{D}_n$ or a subgroup of it. But since Gale's evenness criterion is invariant under rotations and reflections of the star circle, all permutations in $\mathbb{D}_n$ actually yield combinatorial automorphisms of $C_d(n)$. Together this proves $\mathrm{AUT}(C_d(n)) \cong \mathbb{D}_n$ if $n \geq d + 3$ and d is even. Furthermore, $\mathrm{AUT}(C_d(n))$ acts vertex transitively.

If the dimension d is odd the restrictions are even more severe. The pairs of inner vertices form a path of length $n-2$ in the graph of $C_d(n)$ that must be mapped to itself by the induced graph automorphism. Thus, as combinatorial automorphisms of $C_d(n)$ map border vertices to border vertices, $\mathrm{AUT}(C_d(n))$ must be isomorphic to a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, $\mathrm{AUT}(C_d(n))$ does not act vertex transitively in this case.

Reversing the order of all vertices, i.e. the permutation $(1, n)(2, n-1) \ldots$, clearly is an automorphism of $C_d(n)$. To prove $\mathrm{AUT}(C_d(n)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ it thus suffices to show that the transposition $(1, n)$ of the two border vertices is an combinatorial automorphism of $C_d(n)$ as well. Indeed, facets that contain both border vertices stay fixed under this permutation. Facets without any border vertices do not exist, since d is odd. Facets with only one border vertex have one border block, which then is of odd size. The border vertex of this border block is missing after the permutation. Hence the block becomes even. Thus the image of this kind of facet is a facet again.

Geometric Realizations

In this section, we will realize the cyclic polytopes $C_d(n)$ with $n \geq d{+}3$ vertices such that their combinatorial automorphisms are induced by self-congruences, i.e., by orthogonal affine transformations.

If the dimension d is even we can place the vertices $v_1, \ldots, v_n \in \mathbb{R}^d$ on the Carathéodory curve γ_{C} as symmetrically as possible via

$$v_i := \gamma_{\mathrm{C}} \left(\frac{2\pi}{n} \cdot i \right) \ .$$

The combinatorial automorphism group of $C_d(n)$ is isomorphic to $\mathbb{D}_n$. It is generated by the cyclic permutation $(1, 2, \ldots, n)$ and the reflection $(1, n -$

$1)(2,n-2)\ldots$ (fixing n and, if n is even, $\frac{n}{2}$). The cyclic permutation is induced by the linear map defined via the orthogonal matrix

$$\begin{pmatrix} \boxed{\alpha} & & & \\ & \boxed{\alpha^2} & & \\ & & \ddots & \\ & & & \boxed{\alpha^{d/2}} \end{pmatrix} \qquad \text{with} \qquad \alpha = \begin{pmatrix} \cos(\frac{2\pi}{n}) & \sin(\frac{2\pi}{n}) \\ -\sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix} ,$$

and the reflection is obtained from the diagonal matrix

$$\begin{pmatrix} -1 & & & & \\ & +1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & +1 \end{pmatrix} .$$

If the dimension d is odd, then we choose the inner vertices

$$v_2 := \gamma_m(t_2)\ ,\ v_3 := \gamma_m(t_3)\ ,\ \ldots\ ,\ v_{n-1} := \gamma_m(t_{n-1})$$

on the curve defined via

$$\gamma_m(t) = (t, t^2, \ldots, t^{d-1}, 0)$$

with parameters that satisfy $t_i = -t_{n+1-i}$. Their convex hull $C_{\text{inner}} := \operatorname{conv}\{v_2, v_3, \ldots, v_{n-1}\}$ is a cyclic polytope of type $C_{d-1}(n-2)$ located in the subspace $\mathbb{R}^{d-1} \subset \mathbb{R}^d$ defined by $x_d = 0$.

We claim that every hyperplane of $\mathbb{R}^{d-1}$ that defines a facet of C_{inner} intersects the axis $\mathbb{R} \cdot e_{d-1}$ (e_i being the i-th standard unit vector) in a single point. Indeed, for each such hyperplane $H \subset \mathbb{R}^{d-1}$ there are some pairwise different parameters $\tilde{t}_1, \ldots, \tilde{t}_{d-1} \in \{t_2, \ldots, t_{n-1}\}$ such that H is defined via the linear equation $a^T x = a_0$ given by

$$\det \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & \tilde{t}_1 & \tilde{t}_2 & \ldots & \tilde{t}_{d-1} \\ x_2 & \tilde{t}_1^2 & \tilde{t}_2^2 & \ldots & \tilde{t}_{d-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{d-1} & \tilde{t}_1^{d-1} & \tilde{t}_2^{d-1} & \ldots & \tilde{t}_{d-1}^{d-1} \end{pmatrix} = 0$$

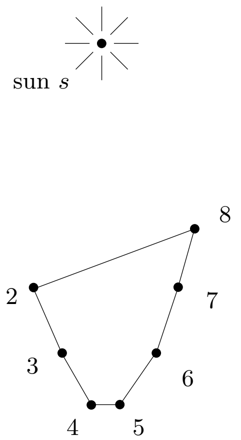
(see [5, Thm. 0.7]). Due to

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \tilde{t}_1 & \tilde{t}_2 & \dots & \tilde{t}_{d-1} \\ \tilde{t}_1^2 & \tilde{t}_2^2 & \dots & \tilde{t}_{d-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{t}_1^{d-2} & \tilde{t}_2^{d-2} & \dots & \tilde{t}_{d-1}^{d-2} \end{pmatrix} \neq 0$$

we have $a^T e_{d-1} \neq 0$, which proves the claim.

Therefore, we can choose $\lambda > 0$ large enough that every facet defining hyperplane of C_{inner} intersects $\mathbb{R} \cdot e_{d-1}$ in a (unique) point $\lambda' \cdot e_{d-1}$ with $\lambda' < \lambda$. We call a facet of C_{inner} *bright* if its defining hyperplane separates (the relative interior of) C_{inner} from $s := s \cdot e_{d-1}$; otherwise it is *dark* (imagine the sun located at s). Equivalently, in the line shelling of C_{inner} induced by the line $s \cdot e_{d-1}$ (oriented from the origin to e_{d-1}), the bright facets are the ones that come before "passing through infinity."

Let F be a facet of C_{inner}, defined by a hyperplane H of $\mathbb{R}^{d-1}$. Let H_{bright} and H_{dark} be the corresponding open halfspaces with $s \in H_{\text{bright}}$. Since γ_m is a curve of degree $d-1$, its intersection with H has cardinality at most d. Thus, these intersection points are precisely the vertices of F. Since $d-1$ is even, the curve γ_m comes from and leaves to H_{bright} (i.e., the intersection of H_{dark} with γ_m is bounded). Consequently, the bright facets are those whose star tables have two odd border blocks, while the dark facets have two even border blocks. We call a ridge *gray* if it is the intersection of a bright facet with a dark facet. A ridge is gray if and only if its star table has one odd and one even border block. Figure 8.4 illustrates the notions.



2	3	4	5	6	7	8
⋆						⋆

A bright facet,

2	3	4	5	6	7	8
		⋆	⋆			

a dark facet,

2	3	4	5	6	7	8
⋆						

and a gray ridge.

Fig. 8.4. Illustration of $C_{\text{inner}} = C_2(7)$.

To construct our desired symmetric realization of the odd dimensional cyclic polytope $C_d(n)$ we "split" s into the remaining two vertices $v_1 := s + e_d$ and $v_n := s - e_d$, and claim that the convex hull

$$C := \operatorname{conv}(C_{\text{inner}} \cup \{v_1, v_n\})$$

is a d-dimensional cyclic polytope (with n vertices).

The polytope C has two kinds of facets: those that contain only one vertex from $\{v_1, v_n\}$, and those that contain both v_1 and v_n. Facets of the first kind are precisely the pyramids over dark facets of C_{inner} (with apex v_1 or v_n), and the facets of the second kind are the convex hulls $\operatorname{conv}(R \cup \{v_1, v_n\})$, where R is a gray ridge of C_{inner}.

It turns out that the star patterns of C obey Gale's evenness criterion, and, conversely, all Gale's evenness patterns appear in this way as facets of C. Hence, C is a realization of $C_d(n)$.

The combinatorial automorphism group $\text{AUT}(C_d(n))$ is generated by the reflection $(1, n)(2, n-1)\ldots$ and by the transposition $(1, n)$, which are induced by the orthogonal linear transformations defined by the diagonal matrices with entries $(-1, +1, \ldots, -1, +1, +1)$ (recall that we have $t_i = -t_{n+1-i}$) and $(+1, +1, \ldots, +1, +1, -1)$, respectively. This concludes the proof of Theorem 8.4.

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