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On rank-perfect subclasses of near-bipartite graphs

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Abstract. Shepherd (1995) proved that the stable set polytopes of near-bipartite graphs are given by constraints associated with the complete join of antiwebs only. For antiwebs, the facet set reduces to rank constraints associated with single antiwebs by Wagler (2004b). We extend this result to a larger graph class, the complements of fuzzy circular interval graphs, recently introduced in Chudnovsky and Seymour (2004).

Key words: Stable set polytope, rank constraints, antiwebs

1 Introduction

The *stable set polytope* STAB(G) of a graph $G = (V, E)$ is defined as the convex hull of the incidence vectors of all stable sets of G (a set $V' \subseteq V$ is stable if the nodes in V' are mutually nonadjacent). A canonical relaxation of STAB(G) is the *clique constraint polytope* QSTAB(G) given by all "trivial" facets, the *nonnegativity constraints*

$$x_i \geq 0 \tag{0}$$

for all nodes i of G and by the *clique constraints*

$$\sum_{i \in Q} x_i \leq 1 \tag{1}$$

for all cliques $Q \subseteq G$ (a set $V' \subseteq V$ is a clique if the nodes in V' are mutually adjacent). We have STAB$(G) \subseteq$ QSTAB(G) for any graph but equality for *perfect* graphs only by Chvátal (1975). According to a famous characterization recently achieved by Chudnovsky et al. (2003), they are precisely the graphs without chordless cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, or their complements, the *odd antiholes* $\overline{C}_{2k+1}$ (the complement $\overline{G}$ has the same node set as G but two nodes are adjacent in $\overline{G}$ iff they are non-adjacent in G); see Ramirez-Alfonsin and Reed (2001) for more information on perfect graphs.

For all imperfect graphs G it follows that $\text{STAB}(G) \subset \text{QSTAB}(G)$. Thus, further constraints are needed to describe their stable set polytopes. A natural way to generalize clique constraints is to investigate *rank constraints*

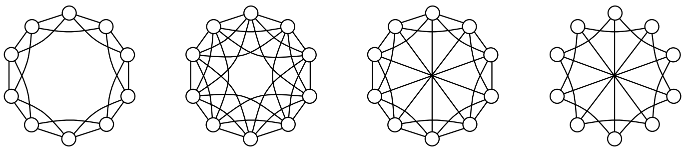
$$\sum_{i \in G'} x_i \leq \alpha(G') \tag{2}$$

associated with *arbitrary* induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the cardinality of a maximum stable set in G' (note $\alpha(G') = 1$ holds iff G' is a clique). For convenience, we also write (2) as $x(G') \leq \alpha(G')$.

A canonical relaxation of $\text{QSTAB}(G)$ is the *rank constraint polytope* $\text{RSTAB}(G)$ given by the nonnegativity constraints (0) and all rank constraints (2); a graph G is *rank-perfect* iff $\text{STAB}(G)$ and $\text{RSTAB}(G)$ coincide, see Wagler (2004a). Every perfect graph is rank-perfect by construction as well as *t-perfect graphs* by Chvátal (1975) resp. *h-perfect graphs* by Grötschel, Lovász, and Schrijver (1988) where rank constraints associated with edges, triangles, and odd holes resp. cliques of arbitrary size and odd holes are allowed only. Further classes of rank-perfect graphs are line graphs by Edmonds (1965) and antiwebs by Wagler (2004b).

A *line graph* is obtained by taking the edges of an original graph as nodes and connecting two nodes iff the original edges are incident; all facets of the stable set polytopes are known from matching theory, see Edmonds (1965), implying that line graphs are rank-perfect.

An *antiweb* $\overline{W}_n^k$ is the complement of a *web* W_n^k: a cycle-symmetric graph with nodes $1, \ldots, n$ where i and j are adjacent if they differ by at most k (see Fig. 1 for examples of webs and antiwebs). Note that antiwebs contain all cliques by $K_n = \overline{W}_n^0$, odd antiholes by $\overline{C}_{2k+1} = \overline{W}_{2k+1}^1$, and odd holes by $C_{2k+1} = \overline{W}_{2k+1}^{k-1}$.



W_{10}^2 W_{10}^3 $\overline{W}_{10}^2$ $\overline{W}_{10}^3$

Fig. 1. Examples of webs and antiwebs

In Wagler (2004b), it is shown that the stable set polytopes of antiwebs are given by nonnegativity constraints and rank constraints associated with cliques and antiwebs only.

Let us call a graph *a-perfect* if its stable set polytope is given by nonnegativity constraints and rank constraints associated with antiwebs only. As such antiweb constraints generalize clique constraints, odd antihole constraints, and odd hole constraints, the class of a-perfect graphs is a common generalization of perfect graphs, t-perfect graphs, h-perfect graphs, and antiwebs.

In this paper, we use a result of Shepherd (1995) on the stable set polytopes of near-bipartite graphs to present a further class of a-perfect graphs: the complements of so-called fuzzy circular interval graphs.

A graph is called *near-bipartite* if the set of nonneighbors of any node is bipartite, i.e., it splits into two stable sets. All antiwebs and the complements of line graphs satisfy this property. The complements of near-bipartite graphs are called *quasi-line*. Chudnovsky and Seymour (2004) introduced recently a subclass of quasi-line graphs, the fuzzy circular interval graphs, obtained as follows.

Let $\mathcal{C}$ be a circle and $\mathcal{I} = \{I_1, \ldots, I_m\}$ be a collection of intervals $I_k = [l_k, r_k]$ in $\mathcal{C}$ s.t. there is no proper containment of intervals in $\mathcal{I}$ and no two intervals share an endpoint. Further, take a finite multiset $V = \{v_1, \ldots, v_n\}$ of points in $\mathcal{C}$ (i.e., $v_i \in \mathcal{C}$ may occur in V with a multiplicity > 1).

The *fuzzy circular interval graph* $G(V, \mathcal{I}) = (V, E_1 \cup E_2')$ has node set V and edge set $E_1 \cup E_2'$ where

$$
\begin{aligned}
E_1 &= \{v_i v_j : \exists I_k \in \mathcal{I} \text{ with } v_i, v_j \in I_k \text{ and } \{v_i, v_j\} \neq \{l_k, r_k\}\} \\
E_2' &\subseteq \{v_i v_j : \exists I_k \in \mathcal{I} \text{ with } v_i = l_k, v_j = r_k\} = E_2.
\end{aligned}
$$

For illustration, look at the circle with multiset V and intervals $\mathcal{I}$ in Fig. 2(a). The resulting sets E_1 and E_2 are depicted in Fig. 2(b) as solid and dashed lines, resp. One possible fuzzy circular interval graph $G(V, \mathcal{I}) = (V, E_1 \cup E_2')$ is shown in Fig. 2(c).

It is easy to check that fuzzy circular interval graphs constitute a subclass of quasi-line graphs and a superclass of webs. Thus, their complements constitute a subclass of near-bipartite graphs and a superclass of antiwebs.

We are going to strengthen the description of the stable set polytopes for near-bipartite graphs obtained by Shepherd (1995) (see Sect. 2) for complements of fuzzy circular interval graphs, establishing their a-perfectness (see Sect. 3).

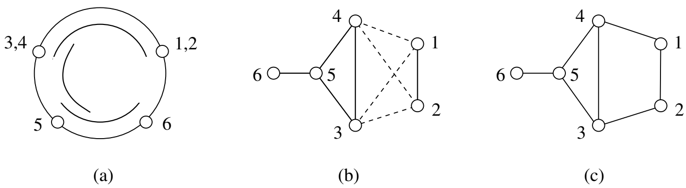


Fig. 2. An Example of a fuzzy circular interval graph

2 On stable set polytopes of near-bipartite graphs

Shepherd (1995) achieved the following description of the stable set polytopes of near-bipartite graphs:

Theorem 1 (Shepherd (1995)). *The only nontrivial facets of stable set polytopes of near-bipartite graphs are constraints*

$$\sum_{i \le k} \frac{1}{\alpha(\overline{W}_i)} x(\overline{W}_i) + x(Q) \le 1 \tag{3}$$

associated with complete joins of prime antiwebs $\overline{W}_1, \ldots, \overline{W}_k$ *and a clique* Q.

The *complete join* of two disjoint graphs G_1 and G_2 is obtained by joining every node of G_1 and every node of G_2 by an edge. An antiweb $\overline{W}_n^{k-1}$ is *prime* if k and n are relatively prime. All odd holes and odd antiholes are prime, the antiweb $\overline{W}_{10}^2$ in Fig. 1 has this property as well, but not $\overline{W}_{10}^3$.

To illustrate Theorem 1, consider the complete join of a C_7 and a $\overline{C}_7$ as near-bipartite graph G. Its stable set polytope has as nontrivial facets

$$\begin{aligned} x(Q) &\le 1 \ \forall \text{ maximal cliques } Q \subseteq G \\ x(\overline{C}_7) + 2x(Q) &\le 2 \ \forall \text{ maximal cliques } Q \subseteq C_7 \\ x(C_7) + 3x(Q) &\le 3 \ \forall \text{ maximal cliques } Q \subseteq \overline{C}_7 \\ 2x(C_7) + 3x(\overline{C}_7) &\le 6 \end{aligned}$$

i.e., constraints (3) associated with either a maximal clique of G, the complete join of one antiweb and a maximal clique of the other antiweb, or the complete join of both antiwebs (coefficients are scaled to be integral).

For subclasses of near-bipartite graphs, the facet sets can admit a simpler structure as in the above theorem only if certain kinds of prime antiwebs or certain complete joins can be excluded.

Shepherd (1995) proved that odd antiholes are the only prime antiwebs occurring in complements of line graphs. Since all odd antiholes have stability number two, all facets of type (3) can be scaled to be 0,1,2-valued in this case.

For the class of antiwebs, the occurrence of certain antiwebs can clearly not be excluded. In Wagler (2004b) it was shown that no antiweb contains the complete join of two prime antiwebs or of a prime antiweb and a nonempty clique, implying that all facets are 0,1-valued.

We extend the latter result to the larger class of complements of fuzzy circular interval graphs, for short called *co-fuzzy circular interval graphs*.

3 Co-fuzzy circular interval graphs are a-perfect

In order to simplify the constraints (3) for co-fuzzy circular interval graphs by excluding complete joins, we discuss whether fuzzy circular interval graphs may admit the disjoint union of a prime web and a single node. This is obviously possible in general quasi-line graphs, as they can even *be* the disjoint union of a prime web and a single node. In addition, there exist connected quasi-line graphs containing such disjoint unions, as indicated in Fig. 3.

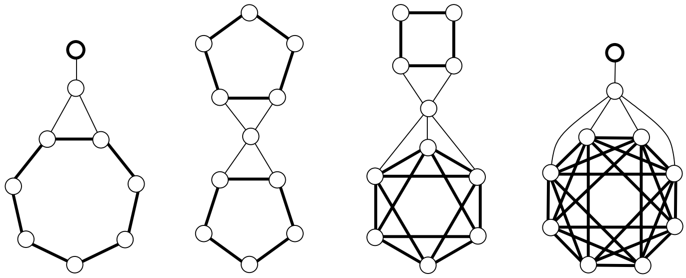


Fig. 3. Disjoint unions of webs in quasi-line graphs

The two right graphs are even fuzzy circular interval involving webs W^k_{2k+2} for $k = 1, 2, 3$. Note that the nodes of any W^k_{2k+2} can be partitioned into two cliques $\{1, \dots, 1+k\}$ and $\{k+2, \dots, 2k+2\}$ allowing a representation by choosing the two cliques as multiple endpoints of one interval (Fig. 2 contains such a representation of W^k_{2k+2} with $k = 1$). We shall ensure that this construction is possible for the webs W^k_{2k+2} only, but not for *prime* webs.

The key tool is to prove that any web W^k_n with $n > 2k + 2$ has precisely *one* representation as fuzzy circular interval graph, namely, the following *canonical* one: distribute the point set $V = \{1, \dots, n\}$ without multiplicities in this order on $\mathcal{C}$ and take a collection $\mathcal{I} = \{I_1, \dots, I_n\}$ of arcs in $\mathcal{C}$ with $I_i \cap V = \{i, \dots, i + k\}$ for $1 \le i \le n$; then $G(V, \mathcal{I})$ obviously equals the web W^k_n. As an example, Fig. 4 shows the canonical representation of W^2_7.

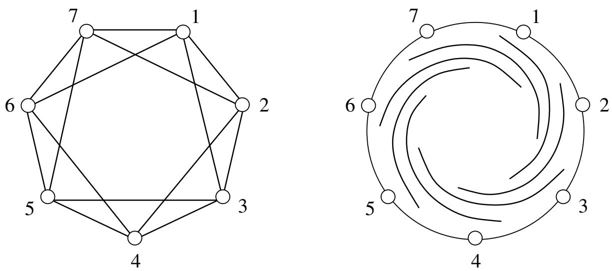


Fig. 4. The canonical representation of W^2_7

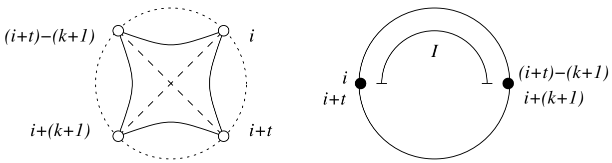


Fig. 5. The case of multiple points

This implies that the intervals representing a web W_n^k with $n > 2k+2$ as subgraph of a fuzzy circular interval graph G already occupy the whole circle $\mathcal{C}$, see Fig. 4. Thus every node in $G - W_n^k$ has a neighbor in W_n^k and no disjoint union of W_n^k and a node in $G - W_n^k$ is possible. (This implies that every W_n^k with $n > 2k+2$ is dominating in G and, in particular, G has to be connected.) Thus, we shall prove:

Lemma 1. *Any web W_n^k with $n > 2k+2$ admits no other representation as fuzzy circular interval graph than the canonical one.*

Proof. Let $G(V, \mathcal{I})$ be a representation of a web W_n^k with $n > 2k+2$ as fuzzy circular interval graph. In order to verify the assertion of the lemma we ensure first that V does not contain multiple points.

Claim. V does not contain points with a multiplicity > 1.

Assume on the contrary, there are multiple points in V and consider, among them, two points i and $i+t$ at maximum distance in W_n^k. Since multiple points are adjacent, we obtain $0 < t \leq k$.

Consider the node $(i+t)-(k+1)$ in W_n^k: by construction, it is adjacent to i but not to $i+t$ (by $n > 2k+2$). Hence, $\mathcal{I}$ contains an interval I having i and $i+t$ as multiple endpoint and $(i+t)-(k+1)$ as opposite endpoint (then we are free of linking i and $(i+t)-(k+1)$ by an edge but $i+t$ and $(i+t)-(k+1)$ not as both edges belong to E_2); see Fig. 5.

Further, consider the node $i+(k+1)$ in W_n^k being adjacent to $i+t$ but not to i. As before, this is possible only if i, $i+t$ and $i+(k+1)$ are opposite endpoints of an interval in $\mathcal{I}$. Since i and $i+t$ cannot be endpoints of two intervals in $\mathcal{I}$, this has to be the same interval I implying that $i+(k+1)$ is a multiple point of $(i+t)-(k+1)$, as shown in Fig. 5. Hence $i+(k+1)$ and $(i+t)-(k+1)$ are adjacent in W_n^k and, by the choice of t, we infer

$$\begin{aligned} (i+t)-(k+1) &\leq i+(k+1)+t \\ i &\leq i+(2k+2) \end{aligned}$$

and $n \leq 2k+2$ follows, yielding a contradiction.

Claim. $G(V, \mathcal{I})$ is the canonical representation of W_n^k.

As Claim 1 shows, adjacencies in W_n^k cannot be realized by multiple points in V but as different points belonging to the same interval in $\mathcal{I}$ only. In particular, consecutive points in V form a clique in W_n^k only if they belong to the same interval in $\mathcal{I}$ (as only other cliques, triangles formed by non-consecutive points are possible). Denote by $Q(i) = \{i, \dots, i+k\}$ the maximum clique of W_n^k starting in node i. Then, obviously, $Q(i)$ and $Q(i+1)$ intersect precisely in the nodes $\{i+1, \dots, i+k\}$. Consequently, there exist intervals $I_i, I_{i+1} \in \mathcal{I}$ with

$$\begin{aligned}(I_i \setminus I_{i+1}) \cap V &= \{i\}\\ (I_i \cap I_{i+1}) \cap V &= \{i+1, \dots, i+k\}\\ (I_{i+1} \setminus I_i) \cap V &= \{i+k+1\}.\end{aligned}$$

Repeating this argumentation for all maximum cliques $Q(1), \dots, Q(n)$ finally yields the assertion. □

Thus, every *prime* web W_n^k has only the canonical representation and is, as mentioned above, as subgraph of a fuzzy circular interval graph dominating. This implies:

Corollary 1. *No fuzzy circular interval graph contains the disjoint union of a prime web and a single node.*

Turning to the complements we, therefore, obtain the nonexistence of the complete join of a prime antiweb and a single node. Thus, the complete join of two prime antiwebs as well as the complete join of a prime antiweb and a nonempty clique are excluded. Hence, the above lemma implies: the only nontrivial facets of the stable set polytope of a co-fuzzy circular interval graph are constraints (3) not consisting of different facet blocks but associated with *either* a prime antiweb *or* a clique. Since both prime antiweb and clique constraints are rank constraints in particular, we obtain:

Theorem 2. *The stable set polytope of a co-fuzzy circular interval graph has as only nontrivial facets rank constraints associated either with cliques or with prime antiwebs.*

As an immediate consequence of Theorem 2 we obtain:

Corollary 2. *Co-fuzzy circular interval graphs are a-perfect.*

Note added in proof. A breakthrough in the area was recently obtained by Eisenbrand et al. (2005) who gave a description of the stable set polytopes of quasi-line graphs, the complements of near-bipartite graphs.

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