On the Complexity of Dissociation Set Problems in Graphs

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Abstract: A subset of vertices in a graph is called a dissociation set if it induces a subgraph with vertex degree at most 1. A dissociation set *D* is maximal if no other dissociation set contains *D*. The complexity of finding a dissociation set of maximum size in line graphs and finding a maximal dissociation set of minimum size in general graphs is considered.

Keywords: Combinatorial mathematics, Graph theory, Dissociation set, Computational complexity

1. INTRODUCTION

Dissociation set problems in graphs find applications in telecommunications and scheduling. The maximum dissociation set problem can be viewed as a generalization both of the maximum induced matching problem which is important in connection with applications to secure communication channels (Golumbic and Lewenstein, 2000) and to the maximum independent set problem which finds applications in manufacturing for production planning and facility location (Haynes *et al.*, 1998).

Let *G* be a graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. For a subset of vertices $X \subseteq V(G)$, the subgraph of *G* induced by *X* is denoted by *G*(*X*). As usual $N_G(x)$, or simply *N*(*x*), denotes the neighborhood of a vertex $x \in V$, i.e., the set of all vertices that are adjacent to *x* in *G*. The *degree* of *x* is defined as $\deg x = |N(x)|$. The maximum vertex degree of *G* is denoted by $\Delta(G)$.

A set $D \subseteq V(G)$ is called a *dissociation set* if it induces a subgraph with a vertex degree at most 1, i.e., $\Delta(G(D)) \leq 1$. A dissociation set *D* is *maximal* if no other dissociation set in *G* contains *D*. Let *DS*(*G*) be the set of all maximal dissociation sets in *G*. Define the *minimum maximal dissociation number* as

$$diss^{-}(G) = \min\{|D| : D \in DS(G)\}$$

and the *maximum dissociation number* as

$$diss^{+}(G) = \max\{|D| : D \in DS(G)\}.$$

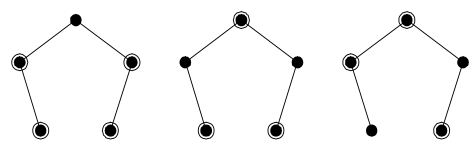

Fig. 1. Maximal dissociation sets of graph P_5.

For example, for the simple path on five vertices (graph $G = P_5$ with $V(P_5) = \{1, 2, 3, 4, 5\}$ and

$E(P_5) = \{12, 23, 34, 45\}$), all maximal dissociation sets (up to symmetry) are shown in Fig. 1 as the sets of encircled vertices. In this case:

$$diss^+(P_5) = 4 \text{ and } diss^-(P_5) = 3 .$$

A *maximum dissociation set* is a dissociation set that contains $diss^+(G)$ vertices. A *minimum maximal dissociation set* is a maximal dissociation set that contains $diss^-(G)$ vertices.

Let us recall some definitions. For a graph *G*, the *line graph* *L*(*G*) is defined as follows:
(*i*) the vertices of *L*(*G*) bijectively correspond to the edges of *G* and
(*ii*) two vertices of *L*(*G*) are adjacent if and only if the corresponding edges of *G* are adjacent.
We denote by G^2 the *square* of graph *G*, i.e., the graph on $V(G)$ in which two vertices are adjacent if and only if they have a distance at most 2 in *G*. Let C_n be a simple cycle on *n* vertices and $K_4 - e$ be a graph obtained from the complete graph on four vertices (K_4) by deleting an edge.

For a set *H* of graphs, a graph *G* is called *H-free* if no induced subgraph of *G* is isomorphic to a graph in *H*. In particular, *claw-free* (or $K_{1,3}$*-free*) graphs constitute a class of graphs that do not contain a *claw* (a star $K_{1,3}$ on four vertices) as an induced subgraph.

2. DISSOCIATION SET PROBLEMS

Consider the two following decision problems associated with the parameters $diss^+(G)$ and $diss^-(G)$. We will refer to these problems as *dissociation set problems*.

MAXIMUM DISSOCIATION SET
Instance: A graph *G* and an integer *k*.
Question: Is $diss^+(G) \geq k$?

MINIMUM MAXIMAL DISSOCIATION SET
Instance: A graph *G* and an integer *k*.
Question: Is $diss^-(G) \leq k$?

The MAXIMUM DISSOCIATION SET problem has been introduced by Yannakakis (1981) and was shown to be NP-complete for the class of bipartite graphs. Boliac *et al*., (2004) strengthen the result of Yannakakis by showing that the problem is NP-complete for bipartite graphs with maximum degree 3 and C_4-free bipartite graphs. It is also known that the problem is NP-complete for planar graphs, see Papadimitriou and Yannakakis (1982).
As mentioned above, dissociation set problems are connected with the well-known independent set and induced matching problems. For a graph *G*, a subset $S \subseteq V(G)$ of vertices is called an *independent set* if no two vertices in *S* are adjacent. In other words, the degrees of all vertices of the subgraph of *G* induced by *S* are equal to 0, i.e., the subgraph is 0-regular. The maximum cardinality of an independent set of *G* is the *independence number*, and it is denoted by $\alpha(G)$.

For a graph *G*, a subset $M \subseteq E(G)$ of edges is called an *induced matching* if
(*i*) set *M* is a matching in *G* (a set of pairwise non-adjacent edges) and
(*ii*) there is no edge in $E(G) \setminus M$ connecting two edges of *M*.

In other words, the degrees of all vertices of the subgraph of *G* induced by the end-vertices of edges of *M* are equal to 1, i.e., the subgraph is 1-regular. The maximum cardinality of an induced matching of *G* is the *induced matching number*, and it is denoted by $\Sigma(G)$.

The decision problems connected with the parameters $\alpha(G)$ and $\Sigma(G)$ can be formulated as follows.

MAXIMUM INDEPENDENT SET
Instance: A graph *G* and an integer *k*.
Question: Is $\alpha(G) \geq k$?

MAXIMUM INDUCED MATCHING
Instance: A graph *G* and an integer *k*.
Question: Is $\Sigma(G) \geq k$?

Notice that the MAXIMUM DISSOCIATION SET problem asks whether in a given graph there exists a maximum induced subgraph with vertex degree equal to 0 or 1, while the MAXIMUM INDEPENDENT SET problem asks whether there exists a maximum induced subgraph with vertex degree equal to 0 and the MAXIMUM INDUCED MATCHING problem asks whether there exists a maximum induced subgraph with vertex degree equal to 1. In addition, $\alpha(G) \leq diss^+(G)$ and $2\Sigma(G) \leq diss^+(G)$ for any graph *G*. In fact, both differences $diss^+(G) - \alpha(G)$ and $diss^+(G) - 2\Sigma(G)$ can be arbitrarily large.

The following table compiles available results on the complexity of the MAXIMUM DISSOCIATION SET problem (MDS), the MAXIMUM INDEPENDENT SET problem (MIS) and the MAXIMUM INDUCED MATCHING problem (MIM) by indicating classes of graphs where the problems are polynomially solvable (P), NP-complete (NP-c) or open (?). For definitions of the graph classes in this table, see Brandstädt *et al*., (1999).

Below it is shown that the MAXIMUM DISSOCIATION SET problem is NP-complete for line graphs and therefore for claw-free graphs.

Table 1. Complexity of MDS, MIM and MIS

Graph classes / Problems	**MDS**	**MIM**	**MIS**
Planar graphs	NP-c	NP-c	NP-c
Triangle free graphs	NP-c	NP-c	NP-c
Bipartite graphs	NP-c	NP-c	P
Claw-free graphs	?	NP-c	P
Line graphs	?	NP-c	P
Chordal graphs	P	P	P
Circular-arc graphs	P	P	P
AT-free graphs	P	P	P

3. COMPLEXITY OF THE MAXIMUM DISSOCIATION SET PROBLEM

An interesting special case of the MAXIMUM DISSOCIATION SET problem arises when the input graph is a line graph. We show that this special case is NP-complete (Theorem 1) by a polynomial time reduction from a variant of the following decision problem.

PARTITION INTO ISOMORPHIC SUBGRAPHS
Instance: Graphs *G* and *H* with $|V(G)| = q|V(H)|$ for some positive integer *q*.
Question: Is there a partition $V_1 \cup V_2 \cup \ldots \cup V_q = V(G)$ such that $G(V_i)$ contains a subgraph isomorphic to *H* for all $i = 1, 2, \ldots, q$?

It is well known that this problem is NP-complete for any fixed *H* that contains a connected component of three or more vertices (Kirkpatrick and Hell, 1978), see also Garey and Johnson (1979).

Consider a special case of PARTITION INTO ISOMORPHIC SUBGRAPHS when *H* is the graph P_3 : the problem PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3. Recall that $P_3 = (u, v, w)$ is a 3-path, i.e., a graph with the edge set $\{uv, vw\}$. Vertex *v* is the central vertex and *u*, *w* are the end-vertices of this 3-path.

The proof of the following statement can be done by a polynomial time transformation from the problem PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3

Theorem 1. MAXIMUM DISSOCIATION SET is an NP-complete problem for line graphs.

This theorem can be strengthened (Theorem 2) using the following results. Orlovich *et al.*, (2008) prove that PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3 is an NP-complete problem for planar bipartite graphs of maximum degree 4 in which every vertex of degree 4 is a cut-vertex.

Theorem 2. MAXIMUM DISSOCIATION SET is an NP-complete problem for planar line graphs of planar bipartite graphs with maximum degree 4.

Obviously, Theorem 2 holds for the class of line graphs of bipartite graphs. This class can be characterized in terms of forbidden induced subgraphs: graph *G* is a line graph of a bipartite graph if and only if *G* does not contain $K_{1,3}$, $K_4 - e$ and C_{2n+1} ($n \geq 2$) as induced subgraphs, see Harary and Holzmann (1974). Theorem 2 implies the following corollary.

Corollary 1. MAXIMUM DISSOCIATION SET is NP-complete for $(K_{1,3}, K_4 - e, C_{2n+1} : n \geq 2)$ -free graphs.

4. SOME POLYNOMIALLY SOLVABLE CASES

Below we show that the MAXIMUM DISSOCIATION SET problem can be solved in polynomial time for some special classes of graphs.

Remind that the graphs H_1 and G_1 shown in Fig. 2 are called *chair* and *bull*, respectively.

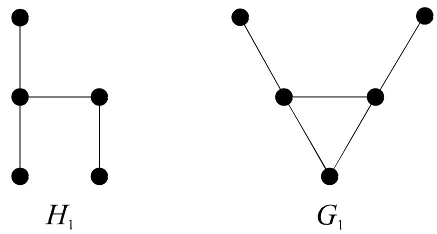

Fig. 2. Graphs H_1 and G_1.

Consider the following construction due to Lozin and Rautenbach (2003). For a graph *G*, let G^* denote the graph with the vertex set $V(G^*) = V(G) \cup E(G)$ such that two vertices $u, v \in V(G^*)$ are adjacent in G^* if and only if either

1. $u, v \in V(G)$ and $uv \in E(G)$ or
2. $u \in V(G)$, $v = xy \in E(G)$ and $N_G(u) \cap \{x, y\} \neq \varnothing$ or
3. $u = xy \in E(G)$, $v = zt \in E(G)$, and $(N_G(x) \cup N_G(y)) \cap \{z, t\} \neq \varnothing$.

An example of graph G^* is shown in Fig. 3 for the graph $G = P_4$. Here $V(P_4) = \{1, 2, 3, 4\}$ and $E(P_4) = \{12, 23, 34\}$.

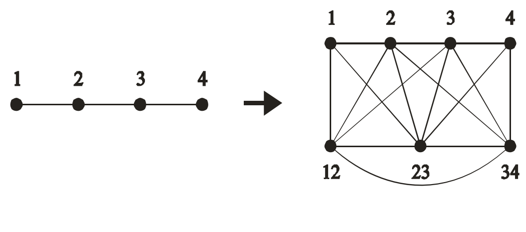

Fig. 3. Graphs P_4 and P_4^*.

Notice that the subgraph of G^* induced by $V(G)$ coincides with G, while the subgraph of G^* induced by $E(G)$ coincides with $(L(G))^2$. Assign to each vertex of G^* in $V(G)$ the weight 1 and to each of the remaining vertices of G^* the weight 2. Then the following statement holds.

Lemma 1 (Lozin and Rautenbach, 2003). An independent set of maximum weight in G^* corresponds to a dissociation set of maximum cardinality $diss^+(G)$ in G.

Using the construction introduced by Lozin and Rautenbach (2003), we prove the following theorem in which H_1 is a chair graph shown in Fig. 2 while the graphs H_2 and H_3 are shown in Fig. 4.

Theorem 3. The graph G^* of a graph G is chair-free if and only if G is (H_1, H_2, H_3) -free.

Based on the results by Alekseev (2004), the following statement has been proved by Lozin and Milanic (2006).

Lemma 2 (Lozin and Milanic, 2008). The maximum weight independent set problem can be solved in polynomial time in the class of chair-free graphs.

Theorem 3 and Lemmas 1 and 2 imply the following result.

Theorem 4. The MAXIMUM DISSOCIATION SET problem can be solved in polynomial time in the class of (H_1, H_2, H_3) -free graphs.

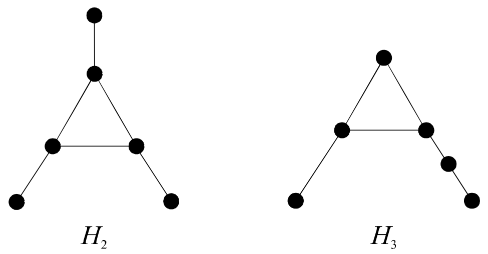

Fig. 4. Graphs H_2 and H_3.

Theorem 4 implies the following corollary.

Corollary 2. The MAXIMUM DISSOCIATION SET problem can be solved in polynomial time in the class of (claw, bull)-free graphs.

For some classes of graphs we can specify the complexity of finding the maximum dissociation number (Theorem 5 and Corollary 3). Remind that a simple path in a graph is called *hamiltonian* if it contains all vertices of the graph.

Theorem 5. Let G be a graph with *n* vertices and containing a hamiltonian path. Then

$$diss^+(L(G)) = \left\lceil \frac{2n}{3} \right\rceil.$$

Corollary 3. The maximum dissociation number can be computed in linear time in the class of line graphs of graphs having a hamiltonian path.

5. THE MINIMUM MAXIMAL DISSOCIATION SET PROBLEM

We establish the complexity of the MINIMUM MAXIMAL DISSOCIATION SET problem by a polynomial time reduction from the following well-known NP-complete decision problem (Garey and Johnson, 1979).

3-SATISFIABILITY
Instance: A collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over a set $X = \{x_1, x_2, \ldots, x_n\}$ of 0-1 variables such that $|c_j| = 3$ for $j = 1, 2, \ldots, m$.
Question: Is there a truth assignment for *X* that satisfies all the clauses in *C*?

Theorem 6. MINIMUM MAXIMAL DISSOCIATION SET is an NP-complete problem.

For some special classes of graphs the problem under consideration can be solved in polynomial time. Using the construction (graph G^*) by Lozin and Rautenbach (2003), we prove the following statement.

Theorem 7. MINIMUM MAXIMAL DISSOCIATION SET is solvable in polynomial time for circular-arc graphs and AT-free graphs.

For the definitions of circular-arc and AT-free graphs, see Brandstädt *et al.* (1999).

6. CONCLUSION

We considered the complexity of finding a dissociation set of maximum size in line graphs and finding a maximal dissociation set of minimum size in general graphs. We have shown that the MAXIMUM DISSOCIATION SET problem is NP-complete for planar line graphs of planar bipartite graphs with maximum degree 4. On the other hand, we have shown that the MAXIMUM DISSOCIATION SET problem can be solved in polynomial time for some special classes of graphs, in particular, for (H_1, H_2, H_3) -free graphs (see Fig. 2 and 4). This class includes (claw, bull)-free graphs as a proper subclass. Moreover, we have shown that the maximum dissociation number can be computed in linear time in the class of line graphs of graphs having a hamiltonian path.

The MINIMUM MAXIMAL DISSOCIATION SET problem has been shown to be NP-complete in the general case and has been shown to be solvable in polynomial time for circular-arc graphs and AT-free graphs. For further research, it is interesting to establish the complexity of the MINIMUM MAXIMAL DISSOCIATION SET problem for chordal graphs.

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