Chapter 10

STABILITY OF OPTIMAL LINE BALANCE WITH GIVEN STATION SET

Yuri N. Sotskov, Alexandre Dolgui, Nadezhda Sotskova, Frank Werner

Abstract: We consider the simple assembly line balancing problem. For an optimal line balance, we investigate its stability with respect to simultaneous independent variations of the processing times of the manual operations. In particular, we prove necessary and sufficient conditions when optimality of a line balance is stable with respect to sufficiently small variations of operation times. We show how to calculate lower and upper bounds for the stability radius, i.e., the maximal value of simultaneous independent variations of operation times with definitely keeping the optimality of line balance.

Key words: assembly line balance, stability analysis.

1. INTRODUCTION

We consider a single-model paced assembly line, which continuously manufactures a homogeneous product in large quantities as in mass production (see [4] for definitions). An assembly line is a sequence of m linearly ordered stations, which are linked by a conveyor belt. A station has to perform the same set of operations repeatedly during the whole life cycle of the assembly line. The set of operations V, which have to be processed by all m stations within one cycle-time c, is fixed. Each operation $i \in V$ is considered as indivisible. All the m stations start simultaneously with the processing of the sequence of their operations and buffers between stations are absent. Technological factors define a partial order on the set of operations, namely, the digraph $G = (V, A)$ with vertices V and arcs A defines a partially ordered set of operations $V = \{1, 2, \ldots, n\}$.

We assume that set V includes operations of two types. More precisely, the non-empty set $\widetilde{V} \subseteq V$ includes all *manual* operations, and the set $V \setminus \widetilde{V}$ includes all *automated* operations. Without loss of generality, we assume that $\widetilde{V} = \{1, 2, \ldots, \widetilde{n}\}$ and $V \setminus \widetilde{V} = \{\widetilde{n}+1, \widetilde{n}+2, \ldots, n\}$ where $1 \leq \widetilde{n} \leq n$. We use the following notations for the vectors of the operation times: $\widetilde{t} = (t_1, t_2, \ldots, t_{\widetilde{n}})$, $\bar{t} = (t_{\widetilde{n}+1}, t_{\widetilde{n}+2}, \ldots, t_n)$ and $t = (\widetilde{t}, \bar{t}) = (t_1, t_2, \ldots, t_n)$.

The *Simple Assembly Line Balancing Problem* is to find an optimal balance of the assembly line for a given number m of stations, i.e., to find a feasible assignment of all operations V to exactly m stations in such a way that the cycle-time c is minimal. In [1, 4], the abbreviation SALBP-2 is used for this problem.

Let the set of operations $V_k^{b_r}$ be assigned to station S_k, $k \in \{1, 2, \ldots, m\}$. Assignment b_r: $V = V_1^{b_r} \cup V_2^{b_r} \cup \ldots \cup V_m^{b_r}$ of operations V to m ordered stations $S_1, S_2, \ldots, S_m$ (where $V_u^{b_r} \cap V_w^{b_r} = \varnothing$, $1 \leq u < w \leq m$) is called a *line balance,* if the following two conditions hold.

1. *Assignment* b_r *does not violate the partial order given by digraph* $G = (V, A)$, *i.e., inclusion* $(i, j) \in A$ *implies that operation* i *is assigned to station* S_k *and operation* j *is assigned to station* S_l *such that* $1 \leq k \leq l \leq m$.

2. *Assignment* b_r *uses exactly* m *stations:* $V_k^{b_r} \neq \varnothing$, $k \in \{1, 2, \ldots, m\}$.

Line balance b_r is *optimal* if along with conditions 1 and 2, it has the minimal cycle-time. We denote the cycle-time for line balance b_r with the vector t of operation times as $c(b_r, t)$:

$$c(b_r, t) = max_{k=1}^{m} \sum_{i \in V_k^{b_r}} t_i \,.$$

Optimality of line balance b_0 with vector t of operation times may be defined as the following condition 3.

3. $c(b_0, t) = min\{c(b_r, t) : b_r \in B\}$, *where* $B = \{b_0, b_1, \ldots, b_h\}$ *is the set of all line balances.*

If $j \in \widetilde{V}$, then the processing time t_j of operation j is a given non-negative real number: $t_j \geq 0$. However, the value of the manual operation time t_j can vary during the life cycle of the assembly line and can even be equal to zero. A zero operation time t'_j means that operation $j \in V_k^{b_r} \cap \widetilde{V}$ is processed by an *additional worker* simultaneously (in parallel) with other operations assigned to station S_k in such a way that the processing of operation j does not increase the station time for S_k:

$$\sum_{i \in V_k^{b_r}} t'_i = \sum_{i \in V_k^{b_r} \setminus \{j\}} t'_i$$

Obviously, the latter equality is only possible if $t'_j = 0$.

If $i \in V \backslash \widetilde{V}$, then operation time t_i is a real number fixed during the whole life cycle of the assembly line. We can assume that $t_i > 0$ for each automated operation $i \in V \backslash \widetilde{V}$. Indeed, an operation with *fixed zero* processing time (if any) has no influence on the solution of SALBP-2, and therefore in what follows, we will consider automated operations, which have only strictly positive processing times.

In contrast to usual *stochastic* problems (see survey [2]), we do not assume any probability distribution known in advance for the random processing times of the manual operations. Moreover, this chapter does not deal with concrete algorithms for constructing an optimal line balance in a stochastic environment. It is assumed that the optimal line balance b_0 is already constructed for the given vector $t = (t_1, t_2, ..., t_n)$ of the operation times. Our aim is to investigate the stability of the optimality of a line balance b_0 with respect to independent variations of the processing times of all manual operations $\widetilde{V} = \{1, 2, \dots, \widetilde{n}\}$ or a portion of the manual operations. More precisely, we investigate the *stability radius* of an optimal line balance b_0, which may be interpreted as the maximum of simultaneous independent variations of the manual operation times with definitely keeping optimality of line balance b_0.

It will be assumed that all operation times t_i, $i \in V$, are real numbers in contrast to the usual assumption that they are integral numbers (see [4]). We need this assumption for the sake of appropriate definitions introduced in Section 2 for a sensitivity analysis. In Section 3, we prove necessary and sufficient conditions for the existence of an *unstable* optimal line balance, i.e., when its stability radius is equal to zero. In Section 4, we show how to calculate the exact value of the stability radius or its upper bound. An algorithm for selecting all stable optimal line balances is discussed in Section 4. Concluding remarks are given in Section 5.

2. DEFINITION OF THE STABILITY RADIUS

The main question under consideration may be formulated as follows. How much can all components of vector $\widetilde{t}$ simultaneously and independently be modified such that the given line balance b_0 remains definitely optimal? To answer this question, we study the notion of the stability radius. The stability radius of an optimal line balance may be defined similarly to the stability radius of an optimal schedule introduced in [5] for a machine scheduling problem. (A survey of known results on sensitivity analysis in machine scheduling is presented in [8].) On the one hand, if the stability radius of line balance b_0 is strictly positive, then any simultaneous independent changes of the operation times t_j, $j \in \widetilde{V}$, within the ball with this radius definitely keep optimality of line balance b_0. On the other hand, if the stability radius of line balance b_0 is equal to zero, then even small changes of the processing times of the manual operations may deprive optimality of line balance b_0.

We consider the space $R^{\tilde{n}}$ of real vectors $\tilde{t} = (t_1, t_2, \ldots, t_{\tilde{n}})$ with the Chebyshev metric. So, the distance $d(\tilde{t}, \tilde{t}^*)$ between vector $\tilde{t} \in R^{\tilde{n}}$ and vector $\tilde{t}^* = (t_1^*, t_2^*, \ldots, t_{\tilde{n}}^*) \in R^{\tilde{n}}$ is calculated as follows:

$$d(\tilde{t}, \tilde{t}^*) = \max\{\, |t_i - t_i^*| : i \in \widetilde{V}\},$$

where $|t_i - t_i^*|$ denotes the absolute value of the difference $t_i - t_i^*$. We also consider the space of non-negative real vectors:

$$R_+^{\tilde{n}} = \{\tilde{t} \in R^{\tilde{n}} : t_i \geq 0,\ i \in \widetilde{V}\}.$$

Let $B(t)$ denote the set of all line balances in the set B, which are optimal for the given vector t of the operation times. The formal definition of the stability radius of an optimal line balance may be given as follows.

Definition 1: *The closed ball* $O_\rho(\tilde{t})$ *in the space* $R^{\tilde{n}}$ *with the radius* $\rho \in R_+^1$ *and the center* $\tilde{t} \in R_+^{\tilde{n}}$ *is called a stability ball of the line balance* $b_0 \in B(t)$, *if for each vector* $t^* = (\tilde{t}^*, \bar{t})$ *of the operation times with* $\tilde{t}^* \in O_\rho(\tilde{t}) \cap R_+^{\tilde{n}}$ *line balance* b_0 *remains optimal. The maximal value of the radius* ρ *of a stability ball* $O_\rho(\tilde{t})$ *of the line balance* b_0 *is called the stability radius denoted by* $\rho_{b_0}(t)$.

In Definition 1, vector $\bar{t} = (t_{\tilde{n}+1}, t_{\tilde{n}+2}, \ldots, t_n)$ of the automated operation times and vector $t = (\tilde{t}, \bar{t}) = (t_1, t_2, \ldots, t_n)$ of all operation times are fixed, while vector $\tilde{t}^* = (t_1^*, t_2^*, \ldots, t_{\tilde{n}}^*)$ of the manual operation times may vary within the intersection of the closed ball $O_\rho(\tilde{t})$ with the space $R_+^{\tilde{n}}$. To illustrate the above notations, we use the following example of SALBP-2.

Let $m = 3$, $\tilde{n} = 2$, $n = 7$ and $t = (\tilde{t}, \bar{t}) = (3, 1, 6, 3, 7, 2, 4)$. Thus, set $\widetilde{V} = \{1, 2\}$ is the set of manual operations, and set $V \setminus \widetilde{V} = \{3, 4, 5, 6, 7\}$ is the set of automated operations. The digraph $G = (V, A)$ and the operation times are represented in Figure 10-1.

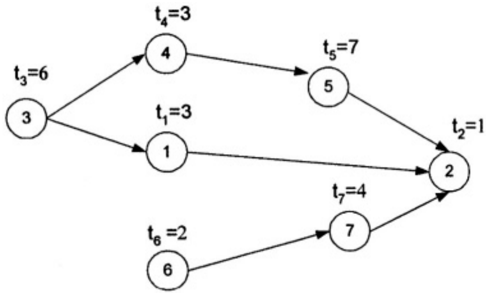

Figure 10-1. Digraph $G = (V, A)$ and operation times

Next, we show that the following line balance b_0:

$V_1^{b_0} = \{3, 4\},\ V_2^{b_0} = \{1, 6, 7\},\ V_3^{b_0} = \{2, 5\}$

is optimal. To this end, we can use the obvious lower bound (1) for the minimal cycle-time.

If all operation times t_i, $i \in V$, are integral numbers, then

$$min\{c(b_r, t) : b_r \in B\} \geq \left\lceil \sum_{i=1}^{n} t_i \Big/ m \right\rceil . \tag{1}$$

Hereafter, $\lceil a \rceil$ denotes the smallest integral number greater than or equal to a. For the above line balance b_0, we have the following equalities:

$$\left\lceil \sum_{i=1}^{n} t_i \Big/ m \right\rceil = \lceil 26/3 \rceil = 9 = c(b_0, t),$$

which imply that b_0 is an optimal line balance since $c(b_0, t)$ is equal to the right-hand side of inequality (1).

Let $\widetilde{V}_k^{b_r}$ denote the subset of all manual operations of set $V_k^{b_r}$. For each optimal line balance $b_r \in B(t)$, we can define a set $W(b_r)$ of all subsets $\widetilde{V}_k^{b_r}$, $k \in \{1, 2, ..., m\}$, such that

$$\sum_{i \in V_k^{b_r}} t_i = c(b_0, t).$$

It should be noted that set $W(b_r)$ may include the empty set as its element. E.g., in the example presented in Figure 10-1 for the optimal line balance $b_0 \in B(t)$, we have $W(b_0) = \{\emptyset, \{1\}\}$ since $\widetilde{V}_1^{b_0} = \emptyset$, $\widetilde{V}_2^{b_0} = \{1\}$, and

$$\sum_{i \in V_1^{b_0}} t_i = \sum_{i \in V_2^{b_0}} t_i = c(b_0, t) = 9.$$

Note that the empty set may be considered as a proper subset of any non-empty set, e.g., we can write $\widetilde{V}_1^{b_0} \subset \widetilde{V}_2^{b_0}$.

3. ZERO STABILITY RADIUS

In this section, we derive necessary and sufficient conditions for the existence of an unstable optimal line balance $b_0 \in B(t)$.

Theorem 1: *Let inequality $t_i > 0$ hold for each manual operation $i \in \widetilde{V}$. Then for line balance $b_0 \in B(t)$, equality $\rho_{b_0}(t) = 0$ holds if and only if there exists a line balance $b_r \in B(t)$ such that condition $W(b_0) \subseteq W(b_r)$ does not hold.*

Proof: *Sufficiency.* Let there exist a line balance $b_r \in B(t)$, for which condition $W(b_0) \subseteq W(b_r)$ does not hold.

Hence, there exists at least one set $\widetilde{V}_k^{b_0} \in W(b_0)$, which does not belong to the set $W(b_r)$. We have to consider the following three possible cases (i), (ii) and (iii).

<u>Case (i)</u>: There exists a set $\widetilde{V}_l^{b_r} \in W(b_r)$ such that $\widetilde{V}_k^{b_0}$ is a proper subset of set $\widetilde{V}_l^{b_r}$, i.e. $\widetilde{V}_k^{b_0} \subset \widetilde{V}_l^{b_r}$ and inequality

$$|\widetilde{V}_k^{b_0}| > |\widetilde{V}_l^{b_r} \setminus \widetilde{V}_k^{b_0}| \tag{2}$$

holds.

From the inclusion $\widetilde{V}_k^{b_0} \subset \widetilde{V}_l^{b_r}$ it follows that set $\widetilde{V}_l^{b_r} \setminus \widetilde{V}_k^{b_0}$ is a non-empty set. Let ε be any arbitrarily small real number such that $\varepsilon > 0$ and $\varepsilon \leq t_i$, $i \in \widetilde{V}$. We can construct the following vector $\widetilde{t}^{\,\varepsilon} = (t_1^\varepsilon, t_2^\varepsilon, \ldots, t_{\widetilde{n}}^\varepsilon)$, where

$$t_i^\varepsilon = \begin{cases} t_i + \varepsilon, & if \quad i \in \widetilde{V}_k^{b_0}, \\ t_i - \varepsilon, & if \quad i \in \widetilde{V}_l^{b_r} \setminus \widetilde{V}_k^{b_0}. \end{cases}$$

For all other manual operations $p \in \widetilde{V} \setminus \widetilde{V}_l^{b_r}$, we set $t_p^\varepsilon = t_p$. Note that due to assumption $t_i > 0$ for each operation $i \in \widetilde{V}$, all components of vector $\widetilde{t}^{\,\varepsilon}$ are non-negative, and therefore $\widetilde{t}^{\,\varepsilon} \in R_+^{\widetilde{n}}$. Inequality (2) implies $\widetilde{V}_k^{b_0} \neq \varnothing$. Therefore, since $t_p^\varepsilon \leq t_p$ for each operation $p \in \widetilde{V} \setminus \widetilde{V}_k^{b_0}$, we obtain

$$c(b_0, t^{\varepsilon}) = \sum_{i \in V_k^{b_0}} t_i^{\varepsilon} = c(b_0, t) + \varepsilon \, | \widetilde{V}_k^{b_0} | \tag{3}$$

where $t^{\varepsilon} = (\widetilde{t}^{\,\varepsilon}, \bar{t}\,)$. Due to inequality (2) and equalities $t_p^{\varepsilon} = t_p$, $p \in \widetilde{V} \setminus \widetilde{V}_l^{b_r}$, we obtain

$$c(b_r, t^{\varepsilon}) = \sum_{i \in V_l^{b_r}} t_i^{\varepsilon} = c(b_r, t) + \varepsilon \, | \widetilde{V}_k^{b_0} | - \varepsilon \, | \widetilde{V}_l^{b_r} \setminus \widetilde{V}_k^{b_0} |. \tag{4}$$

Since set $\widetilde{V}_l^{b_r} \setminus \widetilde{V}_k^{b_0}$ is a non-empty set and $c(b_0, t) = c(b_r, t)$, equalities (3) and (4) imply the strict inequality $c(b_0, t^{\varepsilon}) > c(b_r, t^{\varepsilon})$.

As a result, we conclude that for any arbitrarily small $\varepsilon > 0$ ($\varepsilon \leq t_i$, $i \in \widetilde{V}$), there exists a vector $\widetilde{t}^{\,\varepsilon} \in R_+^{\widetilde{n}}$ such that $d(\widetilde{t}, \widetilde{t}^{\,\varepsilon}) = \varepsilon$ and $c(b_0, t^{\varepsilon}) > c(b_r, t^{\varepsilon})$. Therefore, we obtain $b_0 \notin B(t^{\varepsilon})$. Since vector $\widetilde{t}^{\,\varepsilon}$ may be as close to vector $\widetilde{t}$ as desired, we obtain equality $\rho_{b_0}(t) = 0$ in case (i).

<u>Case (ii):</u> There exists a set $\widetilde{V}_l^{b_r} \in W(b_r)$ such that $\widetilde{V}_k^{b_0}$ is a proper subset of set $\widetilde{V}_l^{b_r}$ and inequality $| \widetilde{V}_k^{b_0} | \leq | \widetilde{V}_l^{b_r} \setminus \widetilde{V}_k^{b_0} |$ holds.

Since $\widetilde{V}_k^{b_0} \subset \widetilde{V}_l^{b_r}$, there exists at least one operation $j \in \widetilde{V}_l^{b_r}$, which does not belong to set $\widetilde{V}_k^{b_0} : j \notin \widetilde{V}_k^{b_0}$. For any arbitrarily small $\varepsilon > 0$ ($\varepsilon \leq t_i$, $i \in \widetilde{V}$), we can construct vector $\widetilde{t}^{(\varepsilon)} = (t_1^{(\varepsilon)}, t_2^{(\varepsilon)}, \ldots, t_{\widetilde{n}}^{(\varepsilon)})$, where

$$t_i^{(\varepsilon)} = \begin{cases} t_i + \varepsilon, \ \mathit{if} \quad i \in \widetilde{V}_k^{b_0}, \\ t_i - \varepsilon, \ \mathit{if} \quad i \in \{j\} \cup \{\widetilde{V} \setminus \widetilde{V}_l^{b_r}\}, \\ t_i, \ \mathit{if} \quad i \in \widetilde{V}_l^{b_r} \setminus \{\widetilde{V}_k^{b_0} \cup \{j\}\}. \end{cases}$$

Since $t_p^{(\varepsilon)} \leq t_p$ for each operation $p \in \widetilde{V} \setminus \widetilde{V}_k^{b_0}$, the following equalities must hold:

$$c(b_0, t^{(\varepsilon)}) = \sum_{i \in V_k^{b_0}} t_i^{(\varepsilon)} = c(b_0, t) + \varepsilon \, | \widetilde{V}_k^{b_0} |, \tag{5}$$

where $t^{(\varepsilon)} = (\widetilde{t}^{(\varepsilon)}, \bar{t}\,)$.

Next, we consider two possible subcases: either $\widetilde{V}_k^{b_0} \neq \varnothing$ or $\widetilde{V}_k^{b_0} = \varnothing$.

If $\widetilde{V}_k^{b_0} \neq \varnothing$, then due to equalities (5) and $c(b_0, t) = c(b_r, t)$, we obtain

$$c(b_r, t^{(\varepsilon)}) = \sum_{i \in V_l^{b_r}} t_i^{(\varepsilon)} = c(b_r, t) + \varepsilon\,|\widetilde{V}_k^{b_0}| - \varepsilon < c(b_0, t^{(\varepsilon)}).$$

If $\widetilde{V}_k^{b_0} = \varnothing$, then we can conclude that set $W(b_r)$ does not contain the empty set as its element. Indeed, if $\widetilde{V}_q^{b_r} = \varnothing$ for some index $q \neq l$, then $\varnothing = \widetilde{V}_k^{b_0} \subseteq \widetilde{V}_q^{b_r}$, which contradicts to the above assumption about the set $\widetilde{V}_k^{b_0}$. Therefore, due to equalities $t_p^{(\varepsilon)} = t_p - \varepsilon$, $p \in \widetilde{V} \setminus \widetilde{V}_l^{b_r}$, we obtain

$$c(b_r, t^{(\varepsilon)}) = \sum_{i \in V_l^{b_r}} t_i^{(\varepsilon)} = \mathrm{c}(\mathrm{b_r}, \mathrm{t}) - \varepsilon. \tag{6}$$

In the case of an empty set $\widetilde{V}_k^{b_0}$, condition (5) turns into equality

$$c(b_0, t^{(\varepsilon)}) = c(b_0, t). \tag{7}$$

From (6) and (7), it follows that $c(b_r, t^{(\varepsilon)}) < c(b_0, t^{(\varepsilon)})$. Thus, using the same arguments for vector $\widetilde{t}^{(\varepsilon)}$ as for vector $\widetilde{t}^{\varepsilon}$ (see case (i)), we conclude that $b_0 \notin B(t^{(\varepsilon)})$, and therefore, $\rho_{b_0}(t) = 0$ in case (ii) as well.

<u>Case (iii):</u> There is no set $\widetilde{V}_l^{b_r} \in W(b_r)$ such that $\widetilde{V}_k^{b_0}$ is a subset of set $\widetilde{V}_l^{b_r}$.

It is clear that $\widetilde{V}_k^{b_0} \neq \varnothing$ (otherwise $\widetilde{V}_k^{b_0} = \varnothing \subseteq \widetilde{V}_l^{b_r} \in W(b_r)$). For any arbitrarily small real $\varepsilon > 0$ ($\varepsilon \leq t_i$, $i \in \widetilde{V}$), we can construct the following vector $\widetilde{t}^{[\varepsilon]} = (t_1^{[\varepsilon]}, t_2^{[\varepsilon]}, \ldots, t_{\widetilde{n}}^{[\varepsilon]}) \in R_+^{\widetilde{n}}$, where

$$t_i^{[\varepsilon]} = \begin{cases} t_i + \varepsilon, & \text{if} \quad i \in \widetilde{V}_k^{b_0}, \\ t_i, & \text{if} \quad i \in \widetilde{V} \setminus \widetilde{V}_k^{b_0}. \end{cases}$$

It is easy to convince that

$$c(b_0, t^{[\varepsilon]}) = \sum_{i \in V_k^{b_0}} t_i^{[\varepsilon]} = c(b_0, t) + \varepsilon\,|\widetilde{V}_k^{b_0}| > c(b_r, t^{[\varepsilon]}).$$

The latter inequality follows from the fact that set $\widetilde{V}_k^{b_0}$ is not contained in any set $\widetilde{V}_l^{b_r} \in W(b_r)$ and $\widetilde{t}_i^{[\varepsilon]} = t_i$ for each operation $i \in \widetilde{V} \setminus \widetilde{V}_k^{b_0}$. Using similar arguments for vector $\widetilde{t}^{[\varepsilon]}$ as for vector $\widetilde{t}^{\varepsilon}$ (see case (i)), we conclude that $b_0 \notin B(t^{[\varepsilon]})$, and therefore, $\rho_{b_0}(t) = 0$ in case (iii) as well.

Necessity: Assume that there does not exist a line balance $b_r \in B(t)$, for which condition $W(b_0) \subseteq W(b_r)$ does not hold.

In other words, either $B(t) = \{b_0\}$ or for any line balance $b_r \in B(t) \setminus \{b_0\}$, condition $W(b_0) \subseteq W(b_r)$ holds. Thus, we have to consider the following two possible cases (j) and (jj).

Case (j): $B(t) = \{b_0\}$.

Let us compare line balance b_0 with an arbitrary line balance $b_s \in B \setminus B(t)$. Since line balance b_s is not optimal for vector t of the operation times, the strict inequality $c(b_s, t) > c(b_0, t)$ must hold. Therefore, for any vector $\widetilde{t}^{\delta} \in R_+^{\widetilde{n}}$ with $d(\widetilde{t}, \widetilde{t}^{\delta}) = \delta > 0$, the opposite inequality $c(b_s, t^{\delta}) < c(b_0, t^{\delta})$ may hold for vector $t^{\delta} = (\widetilde{t}^{\delta}, \bar{t})$ only if

$$\delta > \Delta(b_s) = \frac{c(b_s,t) - c(b_0,t)}{\widetilde{n}}. \tag{8}$$

Indeed, one can overcome the strictly positive difference $c(b_s, t) - c(b_0, t)$ only via changing the processing times t_i of the $\widetilde{n}$ manual operations $i \in \widetilde{V}$. Recall that $\widetilde{n} \geq 1$. Due to bound (8), the desired vector $\widetilde{t}^{\delta} \in R_+^{\widetilde{n}}$ cannot be arbitrarily close to vector $\widetilde{t}$.

Since bound (8) must hold for any non-optimal line balance, we conclude that condition (9) holds for the desired vector $\widetilde{t}^{\delta} \in R_+^{\widetilde{n}}$:

$$d(\widetilde{t}, \widetilde{t}^{\delta}) > \Delta = \min\{\Delta(b_s) : b_s \in B \setminus B(t)\} > 0. \tag{9}$$

As a result we obtain $\rho_{b_0}(t) \geq \Delta > 0$.

Case (jj): $B(t) \setminus \{b_0\} \neq \varnothing$.

Obviously, the lower bound (9) for the distance between vector $\widetilde{t}$ and the desired vector is correct in case (jj) as well. Therefore, we have to compare line balance b_0 only with other optimal line balances.

Let b_r be an arbitrary line balance from the set $B(t) \setminus \{b_0\}$. Due to condition $W(b_0) \subseteq W(b_r)$, there exists a subset $W^*(b_r)$ of the set $W(b_r)$ such that $W(b_0) = W^*(b_r)$.

If there exists an index $k \in \{1, 2, \ldots, m\}$ such that

$$\sum_{i \in V_k^{b_0}} t_i < c(b_0,t), \tag{10}$$

then we set

$$\delta(b_0) = \{c(b_0,t) - \max\{\sum_{i \in V_k^{b_0}} t_i : \tilde{V}_k^{b_0} \notin W(b_0)\}\} / \tilde{n}\,.$$

Due to (10), the strict inequality $\delta(b_0) > 0$ holds.

If $\sum_{i \in V_k^{b_0}} t_i = c(b_0, t)$ for each index $k \in \{1, 2, \ldots, m\}$, then we set

$\delta(b_0) = min\{t_i : i \in \tilde{V}\}$.

We consider an arbitrarily small real number δ, where $0 < \delta \leq \delta(b_0)$ and an arbitrary vector $\tilde{t}^{\delta} \in R_+^{\tilde{n}}$, for which equality $d(\tilde{t}, \tilde{t}^{\delta}) = \delta$ holds. Hereafter, we use the notation

$$t^{\delta}(V') = \sum_{i \in V'} t_i^{\delta}\,, \qquad (11)$$

where t^{δ} denotes a vector the components of which are used in the right-hand side of equality (11). Inequality $\delta \leq \delta(b_0)$ implies

$$c(b_0, t^{\delta}) = \max\{t^{\delta}(V_k^{b_0}) : \tilde{V}_k^{b_0} \in W(b_0)\}, \qquad (12)$$

where $t^{\delta} = (\tilde{t}^{\delta}, \bar{t})$. For any line balance $b_r \in B(t) \setminus \{b_0\}$, we obtain

$$c(b_r, t^{\delta}) = \max\{\max\{t^{\delta}(V_l^{b_r}) : \tilde{V}_l^{b_r} \in W^*(b_r) = W(b_0)\},$$
$$\max\{t^{\delta}(V_q^{b_r}) : \tilde{V}_q^{b_r} \in W(b_r) \setminus W^*(b_r)\}\}. \qquad (13)$$

From (12) and (13), it follows that $c(b_0, t^{\delta}) \leq c(b_r, t^{\delta})$. As a consequence, for any $\delta > 0$ ($\delta \leq \delta(b_0)$), inequality $c(b_0, t^{\delta}) \leq c(b_r, t^{\delta})$ holds for an arbitrary vector $t^{\delta} = (\tilde{t}^{\delta}, \bar{t})$ with distance $d(\tilde{t}, \tilde{t}^{\delta}) = \delta > 0$.

From the latter statement and inequalities (9), it follows that $\rho_{b_0}(t) \geq min\{\Delta, \delta(b_0)\} > 0$. Thus, Theorem 1 is proven.

□

The above proof implies the following corollaries.

Corollary 1: If $B(t) = \{b_0\}$, then $\rho_{b_0}(t) > 0$.

Corollary 2: If $\rho_{b_0}(t) > 0$, then $\rho_{b_0}(t) \geq min\{\Delta, \delta(b_0)\}$.

The latter claim gives a lower bound for a strictly positive stability radius.

4. STABLE OPTIMAL LINE BALANCE

Next, we present an algorithm for selecting the set of stable optimal line balances $B^*(t) \subseteq B(t)$, i.e., all optimal line balances $b_r \in B^*(t)$ with strictly positive stability radii: $\rho_{b_r}(t) > 0$.

Algorithm 1

INPUT: $G=(V, A)$, $t = (\tilde{t}, \bar{t})$.

OUTPUT: Set $B^*(t)$ of all stable optimal line balances.

1. Construct the optimal line balances $B(t) = \{b_0, \ldots, b_{h^*}\}$,
   $0 \leq h^* \leq h$.
2. Construct set $W(b_r)$ for each optimal line balance $b_r \in B(t)$.
   Set $B^*(t) = \emptyset$.
3. DO for r = 0, h^*
   DO for $s = 0, h^*; b_s \neq b_r$
   IF condition $W(b_r) \subseteq W(b_s)$ does not hold, THEN GOTO 5.
   IF $b_s = b_{h^*}$, THEN GOTO 4.
   END
4. Line balance b_r is stable: $\rho_{b_r}(t) > 0$.
   Set $B^*(t) := B^*(t) \cup \{b_r\}$. GOTO 6.
5. Line balance b_r is unstable: $\rho_{b_r}(t) = 0$.
6. END

Due to Theorem 1, all stable optimal line balances are selected by Algorithm 1. Within step 1 of Algorithm 1, one has to solve problem SALBP-2 which is binary NP-hard even if $m = 2$ and $A = \emptyset$. The latter claim may be easily proven by a polynomial reduction of the NP-complete *partition* problem to SALBP-2 with $m = 2$ (see e.g. [4]). To reduce the calculations in steps 2 – 6, we can consider a proper subset of set $B(t)$ instead of the whole set.

Returning to the example of problem SALBP-2 presented in Section 2 (see Figure 10-1), we can construct the set $B^*(t) = \{b_0, b_1, b_2\}$ of all optimal line balances, where

$V_1^{b_1} = \{1, 3\}$, $V_2^{b_1} = \{4, 6, 7\}$, $V_3^{b_1} = \{2, 5\}$;

$V_1^{b_2} = \{3, 4\}$, $V_2^{b_2} = \{5, 6\}$, $V_3^{b_2} = \{1, 2, 7\}$.

We find the sets $W(b_1) = \{\emptyset, \{1\}\}$ and $W(b_2) = \{\emptyset\}$ since $\widetilde{V}_1^{b_1} = \{1\}$, $\widetilde{V}_2^{b_1} = \emptyset$ and $\widetilde{V}_1^{b_2} = \widetilde{V}_2^{b_2} = \emptyset$. Due to Theorem 1, we obtain equality $\rho_{b_0}(t) = 0$ since condition $W(b_0) \subseteq W(b_2)$ does not hold for line balance $b_2 \in B(t)$. Similarly, due to Theorem 1, $\rho_{b_1}(t) = 0$ since condition $W(b_1) \subseteq W(b_2)$ does not hold. The only optimal line balance with a strictly positive stability radius is line balance b_2. Indeed, for any optimal line balance $b_r \in B(t)$, condition $W(b_2) \subseteq W(b_r)$ holds. Thus, Algorithm 1 gives the singleton $B^*(t) = \{b_2\}$.

Next, we show how to use Theorem 1 for the calculation of the exact value of a strictly positive stability radius $\rho_{b_s}(t)$ for line balance $b_s \in B^*(t)$. For calculating $\rho_{b_s}(t)$, we have to find a line balance $b_r \in B$ and a vector $\widetilde{t}' = (t'_1, t'_2, \ldots, t'_{\widetilde{n}}) \in R_+^{\widetilde{n}}$ such that

$$c(b_r, t') < c(b_s, t'), \tag{14}$$

where $t' = (\widetilde{t}', \bar{t})$ and vector $\widetilde{t}'$ is the closest vector to vector $\widetilde{t}$, for which inequality (14) holds.

Since value $c(b_r, t)$ linearly depends on the components of vector $\widetilde{t}$, before reaching inequality (14) via a continuous change of the components of vector $\widetilde{t}$, we first reach equality $c(b_r, t^r) = c(b_s, t^r)$ for some new vector $t^r = (\widetilde{t}^r, \bar{t})$, for which the optimal line balance b_s becomes not stable, i.e., equality (15) holds:

$$\rho_{b_s}(t^r) = 0. \tag{15}$$

Let $W(b_r, t^r)$ denote the set of all subsets $\widetilde{V}_k^{b_r}$, $k \in \{1, 2, \ldots, m\}$, with the valid equality

$$t^r(V_k^{br}) = c(b_0, t^r). \tag{16}$$

Due to equality (15) (see Theorem 1), there exists a line balance $b_r \in B(t^r)$ such that condition $W(b_s, t^r) \subseteq W(b_r, t^r)$ does not hold. Therefore, using the same arguments as in the sufficiency proof of Theorem 1 (see cases (i), (ii) and (iii)), we can construct a vector $\widetilde{t}'$ for which inequality (14) holds and $d(\widetilde{t}', \widetilde{t}^r) = \varepsilon > 0$, where ε may be chosen as small as desired.

Thus, the calculation of $\rho_{b_s}(t)$ for line balance $b_s \in B^*(t)$ is reduced to the construction of the closest vector t^r to vector t for which equality (15) holds.

Next, we demonstrate this construction for the example shown in Figure 10-1. Namely, we consider two possibilities (see case (I) and case (II)) how we can reach equality (15) for line balance $b_s = b_2 \in B(t) = \{b_0, b_1, b_2\}$.

Case (I): $b_r \in B(t)$

Since $\rho_{b_2}(t') > 0$, condition

$$W(b_2, t) \subseteq W(b_r, t) \tag{17}$$

holds for any optimal line balance $b_r \in B(t)$ (see Theorem 1). In order to get equality (15) we have to violate a condition like (17), namely, for a new vector $t^r = (\tilde{t}^r, \bar{t})$ condition

$$W(b_2, t^r) \subseteq W(b_r, t^r) \tag{18}$$

must be incorrect. To violate condition (18), we can include a new element into set $W(b_2)$ or delete corresponding elements from the set $W(b_r)$. The latter possibility cannot be realized since set $W(b_r)$ includes the empty set as its element. Therefore, we only can include a new element into the set $W(b_2)$.

It is clear that the only candidate for such an inclusion is the subset $\tilde{V}_3^{b2} = \{1, 2\}$ of set $V_3^{b_2} = \{1, 2, 7\}$. If we set $t_2^r = t_2 + 1 = 2$ and $t_1^r = t_1$, then we obtain $W(b_2, t^r) = W(b_2, t) \cup \{\tilde{V}_3^{b2}\}$, $W(b_2, t^r) = \{\emptyset, \{1, 2\}\}$, and condition (18) does not hold. Therefore, $\rho_{b2}(t^r) \geq d(t, t^r) = 1$.

Case (II): $b_r \in B \setminus B(t)$

In this case, we have to make line balance $b_r \in B \setminus B(t)$ optimal for a new vector $t^r = (\tilde{t}^r, \bar{t})$ violating condition (18). It is easy to see that line balance $b_3 \in B \setminus B(t)$:

$V_1^{b3} = \{3, 6\}$, $V_2^{b3} = \{1, 4, 7\}$, $V_3^{b3} = \{2, 5\}$

may be included into set $B(t^r)$ with a cycle time equal to 9. To this end, we can set $t_1^r = t_1 - 1 = 2$, $t_2^r = t_2$, and obtain $B(t^r) = B(t) \cup \{b_3\}$.

Thus, in both cases (I) and (II), we have $\rho_{b_2}(t^r) \geq d(t, t^r) = 1$. It is easy to convince that vectors $\tilde{t}^r$ constructed in case (I) and case (II) are the closest to vector $\tilde{t}$ with equality (15) being correct. Therefore, due to Theorem 1 we obtain $\rho_{b_2}(t) = 1$.

The same cases (I) and (II) have to be considered for calculating the stability radius for any line balance $b_s \in B^*(t)$. In case (I), we have to compare line balance b_s with all line balances $b_r \in B(t)$ and calculate the following upper bound for the stability radius $\rho_{b_s}(t)$:

$$\max\left\{ \frac{c(b_s,t)-t(V_k^{b_s})}{|\widetilde{V}_u^{b_s} \oplus \widetilde{V}_k^{b_r}|} : |\widetilde{V}_u^{b_s} \oplus \widetilde{V}_k^{b_r}| \geq 1, \widetilde{V}_u^{b_s} \bigcap \widetilde{V}_k^{b_r} \neq \varnothing \right\},$$

where the sign $\oplus$ denotes the direct sum of two sets.

In case (II), we have to compare line balance b_s with line balances $b_r \in B \setminus B(t)$ and calculate the following upper bound for the stability radius $\rho_{b_s}(t)$:

$$\max\left\{ \frac{t(V_k^{b_r})-c(b_s,t)}{|\widetilde{V}_u^{b_s} \oplus \widetilde{V}_k^{b_r}|} : |\widetilde{V}_u^{b_s} \oplus \widetilde{V}_k^{b_r}| \geq 1, \widetilde{V}_u^{b_s} \bigcap \widetilde{V}_k^{b_r} \neq \varnothing \right\}.$$

If all competitive line balances will be compared with line balance $b_s \in B^*(t)$, then we calculate the exact value of $\rho_{b_s}(t)$, otherwise we obtain an upper bound for the stability radius. In order to restrict the set of line balances which have to be compared with b_s, we can use an approach similar to the one derived in [7] for the stability radius of an optimal schedule.

5. CONCLUSION

We can give two remarks how to restrict the set of optimal line balances considered in Algorithm 1. First, it should be noted that we do not distinguish line balances which have only different orders of the subsets V_k^b, $k \in \{1, 2, ..., m\}$, but their set of subsets $\{V_1^b, V_2^b, ..., V_m^b\}$ is the same. Second, in practice not all optimal line balances are suitable for a realization since not only precedence constraints defined by the arc set A have to be taken into account. Therefore, the cardinality of set $B(t)$ used in Algorithm 1 may be essentially smaller than $|B(t)|$.

It is easy to show that SALBP-2 may be considered as the problem of scheduling n partially ordered jobs on m parallel (identical) machines with the makespan criterion. In [3], this problem is denoted as $P \mid prec \mid C_{\max}$. Therefore, the above results for an optimal line balance may be interpreted as results on the stability analysis of an optimal schedule for problem $P \mid prec \mid C_{\max}$.

At the stage of the design of the assembly line, another mathematical problem (denoted as SALBP-1) has to be considered. Problem SALBP-1 is to minimize the number of stations when the cycle-time is given and fixed. The stability of feasibility and optimality of a line balance for problem SALBP-1 have been considered in [6].

ACKNOWLEDGEMENTS

This research was supported by INTAS (Project 00-217) and Fundamental Research Found of Belarus (Project Ф03МС-039).

REFERENCES

1. Baybars I., 1986. A survey of exact algorithms for the simple assembly line balancing problem, *Management Science,* 32, 909-932.
2. Erel E., Sarin S.C., 1998. A survey of the assembly line balancing procedures, *Production Planning & Control,* 9, 414 – 434.
3. Lawler E.L., Lenstra J.K., Rinnooy Kan A.H.G., Shmoys D.B., 1993. Sequencing and scheduling: algorithms and complexity, in: *Handbook in Operations Research and Management Science 4: Logistic of Production and Inventory,* edited by Graves S.C., Rinnooy Kan A.H.G., and Zipkin P., North-Holland, 445-522.
4. Scholl A. 1999. *Balancing and Sequencing of Assembly Lines,* Heidelberg: Physica-Verlag, A Springer-Verlag Company.
5. Sotskov Yu.N., 1991. Stability of an optimal schedule, *European Journal of Operational Research,* 55, 91-102.
6. Sotskov Yu.N., Dolgui A., 2001. Stability radius of the optimal assembly line balance with fixed cycle time, in: *Proceedings of the IEEE Conference ETFA '2001,* 623-628.
7. Sotskov Yu.N., Sotskova N., Werner F., 1997. Stability of an optimal schedule in a job shop', *Omega. International Journal of Management Sciences,* 25, 397-414.
8. Sotskov Yu.N., Tanaev V.S., Werner F., 1998. Stability radius of an optimal schedule: A survey and recent developments, Chapter in: G. Yu (Ed.), *Industrial Applications of Combinatorial Optimization,* 16, Kluwer Academic Publishers, Boston, MA, 72-108.