EXERCISES CHAPTER 5

1. Use the substitution rule for finding the following indefinite integrals:

(a) \( \int e^{\sin x} \cos x \, dx \);
(b) \( \int \frac{\ln x}{x} \, dx \);
(c) \( \int \frac{5}{1-4x} \, dx \);
(d) \( \int \frac{dx}{e^{3x}-2x} \);
(e) \( \int \frac{x \, dx}{\sqrt{x^2 + 1}} \);
(f) \( \int \frac{x^3 \, dx}{\sqrt{1 + x^2}} \);
(g) \( \int \frac{e^{2x} - 2e^x}{e^{2x} + 1} \, dx \);
(h) \( \int \frac{dx}{\sqrt{2 - 9x^2}} \);
(i) \( \int \frac{\cos^3 x \, dx}{\sin^2 x} \);
(j) \( \int \frac{dx}{1 - \cos x} \);
(k) \( \int \frac{dx}{2\sin x + \sin 2x} \).

2. Use integration by parts to find the following indefinite integrals:

(a) \( \int x^2 e^x \, dx \);
(b) \( \int e^x \cos x \, dx \);
(c) \( \int \frac{x}{\cos^2 x} \, dx \);
(d) \( \int \cos^2 x \, dx \);
(e) \( \int x^2 \ln x \, dx \);
(f) \( \int \ln(x^2 + 1) \, dx \).

3. Evaluate the following definite integrals:

(a) \( \int_{-1}^{2} x^2 \, dx \);
(b) \( \int_{0}^{4} \frac{dx}{1 + \sqrt{x}} \);
(c) \( \int_{0}^{\frac{\pi}{2}} \sin^3 x \, dx \);
(d) \( \int_{0}^{4} \frac{x \, dx}{\sqrt{1 + 2x}} \);
(e) \( \int_{0}^{t} \frac{dx}{2x - 1}, \quad t < \frac{1}{2} \);
(f) \( \int_{0}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx \);
(g) \( \int_{-1}^{0} \frac{dx}{x^2 + 2x + 2} \).

4. A firm intends to precalculate the development of cost, sales and profit of a new product for the first five years after launching it. The calculations are based on the following assumptions:

- \( t \) denotes the time (in years) from the introduction of the product beginning with \( t = 0 \);
- \( C(t) = 1,000 \cdot \left[ 4 - (2e^t)/(e^t + 1) \right] \) is the cost as a function of \( t \in \mathbb{R}, t \geq 0 \);
- \( S(t) = 10,000 \cdot t^2 \cdot e^{-t} \) are the sales as a function of \( t \in \mathbb{R}, t \geq 0 \).

(a) Calculate total cost, total sales and total profit for a period of four years.
(b) Find average sales per year and average cost per year for this period.
(c) Find the total profit as a function of the time \( t \).
5. (a) Find
\[ \int_{0}^{2\pi} \sin x \, dx \]
and compute the area enclosed by function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(x) = \sin x \) and the \( x \)-axis.

(b) Compute the area enclosed by the two functions \( f_1 : \mathbb{R} \to \mathbb{R} \) and \( f_2 : \mathbb{R} \to \mathbb{R} \) given by
\[ f_1(x) = x^3 - 4x \quad \text{and} \quad f_2(x) = 3x + 6. \]

6. The throughput \( q = q(t) \) (output per time unit) of a continuously working production plant is given by a function depending on time \( t \):
\[ q(t) = q_0 \cdot \left[ 1 - \left( \frac{t}{10} \right)^2 \right]. \]
The throughput decreases for \( t = 0 \) up to \( t = 10 \) from \( q_0 \) to 0. One overhaul during the time interval \([0, T]\) with \( T < 10 \) effects that the throughput goes up to \( q_0 \). After that it decreases like before.

(a) Graph the function \( q \) with regard to the overhaul.

(b) Let \( t_0 = 4 \) be the time of overhaul. Find the total output for the time interval \([0, T]\) with \( T > 4 \).

(c) Determine the time \( t_0 \) of overhaul which maximizes the total output in the interval \([0, T]\).

7. Determine the following definite integral numerically:
\[ \int_{0}^{1} \frac{dx}{1 + x^2}. \]

(a) Use approximation by trapeziums with \( n = 10 \).

(b) Use Kepler’s formula.

(c) Use Simpson’s formula with \( n = 10 \).

Compare the results of (a), (b) and (c) with the exact value.

8. Evaluate the following improper integrals:

(a) \( \int_{-\infty}^{0} e^x \, dx \);

(b) \[ \int_{1}^{\infty} \frac{dx}{x^2 + 2x + 1}; \]

(c) \[ \int_{0}^{\infty} e^{-\lambda x} \, dx; \]

(d) \[ \int_{0}^{\infty} \lambda x^2 e^{-\lambda x} \, dx; \]

(e) \[ \int_{0}^{4} \frac{dx}{\sqrt{x}; \]

(f) \[ \int_{0}^{6} \frac{2x - 1}{(x + 1)(x - 2)} \, dx. \]
9. Let function \( f \) with
\[
f(t) = 20t + 200
\]
(in EUR) describe the annual rate of an income flow at time \( t \) continuously received over the years from time \( t = 0 \) to time \( t = 6 \). Interest is compounced continuously at a rate of 4 per cent p.a. Evaluate the present value at time zero.

10. Given are a demand function \( D \) and a supply function \( S \) depending on price \( p \) as follows:
\[
D(p) = 12 - 2p \quad \text{and} \quad S(p) = \frac{8}{7} p - \frac{4}{7}.
\]
Find the equilibrium price \( p^* \) and evaluate customer surplus \( CS \) and producer surplus \( PS \). Illustrate the result graphically.