EXERCISES CHAPTER 6

1. Given are the vectors 
\[ \mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -4 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}. \]

(a) Find vectors \( \mathbf{a} + \mathbf{b} - \mathbf{c} \), \( \mathbf{a} + 3\mathbf{b} \), \( \mathbf{b} - 4\mathbf{a} + 2\mathbf{c} \), \( \mathbf{a} + 3(\mathbf{b} - 2\mathbf{c}) \).

(b) For which of the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) do the relations > or \( \geq \) hold?

(c) Find the scalar products \( \mathbf{a}^T \cdot \mathbf{b}, \mathbf{a}^T \cdot \mathbf{c}, \mathbf{b}^T \cdot \mathbf{c}. \) Which of the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are orthogonal? What is the angle between the vectors \( \mathbf{b} \) and \( \mathbf{c} \)?

(d) Compute vectors \( (\mathbf{a}^T \cdot \mathbf{b}) \cdot \mathbf{c} \) and \( \mathbf{a} \cdot (\mathbf{b}^T \cdot \mathbf{c}) \).

(e) Compare number \( |\mathbf{b} + \mathbf{c}| \) with number \( |\mathbf{b}| + |\mathbf{c}| \) and number \( |\mathbf{b}^T \cdot \mathbf{c}| \) with number \( |\mathbf{b}| \cdot |\mathbf{c}| \).

2. Find \( \alpha \) and \( \beta \) so that vectors 
\[ \mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ \alpha \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} \beta \\ 4 \\ -2 \end{pmatrix} \]
are orthogonal.

3. (a) What is the distance between the following points: \((1, 2, 3)\) and \((4, -1, 2)\) in the 3-dimensional Euclidean space \( \mathbb{R}^3 \)?

(b) Illustrate the following sets of points in \( \mathbb{R}^2 \): \( \mathbf{a} \geq \mathbf{b} \) and \( |\mathbf{a}| \geq |\mathbf{b}| \).

4. Given are the vectors 
\[ \mathbf{a}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{a}^2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

Find out which of the vectors 
\[ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \]
are linear combinations of \( \mathbf{a}^1 \) and \( \mathbf{a}^2 \). Is one of the above vectors a convex combination of vectors \( \mathbf{a}^1 \) and \( \mathbf{a}^2 \)? Graph all these vectors.

5. Given are the vectors 
\[ \mathbf{a}^1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \mathbf{a}^2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \mathbf{a}^3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{a}^4 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \]

Show that vector \( \mathbf{a}^4 \) can be expressed as a convex linear combination of vectors \( \mathbf{a}^1, \mathbf{a}^2 \) and \( \mathbf{a}^3 \). Find the convex combinations of vectors \( \mathbf{a}^1, \mathbf{a}^2 \) and \( \mathbf{a}^3 \) graphically.
6. Are the vectors
\[ a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \text{ and } a_3 = \begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix} \]
linearly independent?

7. Do the two vectors
\[ a_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ and } a_2 = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \]
span the 2-dimensional space? Do they constitute a basis? Graph the vectors and illustrate their linear combinations.

8. Do the vectors
\[ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \]
constitute a basis in \( \mathbb{R}^4 \)?

9. Let vectors
\[ a_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } a_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \]
call a basis in \( \mathbb{R}^3 \).

(a) Express vector
\[ a = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} \]
as a linear combination of the three vectors \( a_1, a_2 \) and \( a_3 \) above.

(b) Find all other bases for the 3-dimensional space which include vector \( a \) and vectors from the set \( \{ a_1, a_2, a_3 \} \).

(c) Express vector
\[ b = 2a_1 + 2a_2 + 3a_3 = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \]
by the basis vectors \( a_1, a_2 \) and \( a \).