EXERCISES CHAPTER 7

1. Given are the matrices

\[ A = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & -2 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 0 \end{pmatrix}; \]

\[ C = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 2 & 2 \end{pmatrix}. \]

(a) Find the transposes. Check whether some matrices are equal.
(b) Calculate \( A + D, A - D, A^T - B \) and \( C - D \).
(c) Find \( A + 3(B^T - 2D) \).

2. Find a symmetric and an antisymmetric matrix so that their sum is equal to

\[ A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -2 \end{pmatrix}. \]

3. Calculate all defined products of matrices \( A \) and \( B \):

(a) \( A = \begin{pmatrix} 4 & 2 \\ 3 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 3 & 5 & -1 \\ 7 & 6 & 2 \end{pmatrix}; \)

(b) \( A = \begin{pmatrix} 4 & 3 & 5 \\ 2 & 5 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 2 & 3 \\ 4 & 5 & 2 \\ 3 & 4 & 5 \end{pmatrix}; \)

(c) \( A = \begin{pmatrix} 2 & 3 & 4 & 5 \end{pmatrix}; \quad B = \begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix}; \)

(d) \( A = \begin{pmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ -1 & -2 \end{pmatrix}; \)

(e) \( A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \)

4. Use the matrices given in Exercise 7.3 (d) and verify the equalities

\[ A^T B^T = (BA)^T \quad \text{and} \quad B^T A^T = (AB)^T. \]

5. Given are the matrices

\[ A = \begin{pmatrix} 1 & 0 \\ 7 & -4 \\ 5 & 3 \end{pmatrix}; \quad B = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 7 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix}. \]
(a) Find the dimension of a product of all three matrices if possible.

(b) Test the associative law of multiplication with the given matrices.

6. Calculate all powers of the following matrices:

(a) \[ A = \begin{pmatrix} 0 & 2 & 8 & 1 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \]

(b) \[ B = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}. \]

7. A firm produces by means of two raw materials \( R_1 \) and \( R_2 \) three intermediate products \( S_1, S_2 \) and \( S_3 \), and with these intermediate products two final products \( F_1 \) and \( F_2 \). The numbers of units of \( R_1 \) and \( R_2 \), which are necessary for the production of one unit of \( S_j, j \in \{1,2,3\} \), and the numbers of units of \( S_1, S_2 \) and \( S_3 \), which are necessary for one unit of \( F_1 \) and \( F_2 \), are given in the following tables:

\[
\begin{align*}
| & S_1 & S_2 & S_3 | & | F_1 & F_2 | \\
R_1 & 2 & 3 & 5 & 6 & 0 \\
R_2 & 5 & 4 & 1 & 1 & 4 \\
S_1 & 3 & 2 & 3 & 2 \\
S_2 & 1 & 4 & 1 & 4 \\
S_3 & 6 & 0 & 6 & 0 \\
\end{align*}
\]

Solve the problem by means of matrix operations.

(a) How many raw materials are required when 1,000 units of \( F_1 \) and 2,000 units of \( F_2 \) have to be produced?

(b) The costs for one unit of raw material are 3 EUR for \( R_1 \) and 5 EUR for \( R_2 \). Calculate the costs for intermediate and final products.

8. Given is the matrix

\[ A = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 2 & 4 \\ 0 & 5 & 1 \end{pmatrix}. \]

(a) Find the submatrices \( A_{12} \) and \( A_{22} \).

(b) Calculate the minors \( |A_{12}| \) and \( |A_{22}| \).

(c) Calculate the cofactors of the elements \( a_{11}, a_{21}, a_{31} \) of matrix \( A \).

(d) Evaluate the determinant of matrix \( A \).

9. Evaluate the following determinants:

(a) \[ \begin{vmatrix} 2 & 1 & 6 \\ -1 & 0 & 3 \\ 3 & 2 & 9 \end{vmatrix}; \]

(b) \[ \begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{vmatrix}; \]

(c) \[ \begin{vmatrix} 1 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 0 \end{vmatrix}; \]

(d) \[ \begin{vmatrix} 2 & 7 & 4 & 1 \\ 3 & 1 & 4 & 0 \\ 5 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix}; \]

(e) \[ \begin{vmatrix} -1 & 2 & 4 & 3 \\ 2 & -4 & -8 & -6 \\ 7 & 1 & 5 & 0 \\ 1 & 5 & 0 & 1 \end{vmatrix}; \]

(f) \[ \begin{vmatrix} 3 & 3 & 3 & \cdots & 3 & 3 \\ 3 & 0 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 3 & 3 & 3 & \cdots & 3 & 0 \end{vmatrix}_{(n,n)}. \]
10. Find the solutions $x$ of the following equations:
   
   \[
   \begin{align*}
   (a) \quad & \left| \begin{array}{ccc} -1 & x & x \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{array} \right| = 27; \\
   (b) \quad & \left| \begin{array}{ccc} x & 1 & 2 \\ 3 & x & -1 \\ 4 & x & -2 \end{array} \right| = 2.
   \end{align*}
   \]

11. Find the solution of the following systems of equations by Cramer's rule:
   
   \[
   \begin{align*}
   2x_1 + 4x_2 + 3x_3 &= 1 \\
   3x_1 - 6x_2 - 2x_3 &= -2 \\
   -5x_1 + 8x_2 + 2x_3 &= 4
   \end{align*}
   \]

12. Let $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear mapping described by matrix
   
   \[
   A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 2 & 4 \\ 4 & 1 & 5 \end{pmatrix}.
   \]

   Find the kernel of this mapping.

13. Given are three linear mappings described by the following systems of equations:
   
   \[
   \begin{align*}
   u_1 &= v_1 - v_2 + v_3 \\
   u_2 &= 2v_1 - v_2 - v_3 \\
   u_3 &= -v_1 + v_2 + 2v_3, \\
   v_1 &= -w_1 + w_3 \\
   v_2 &= w_1 + 2w_2 - w_3 \\
   v_3 &= w_2 - 2w_3, \\
   w_1 &= x_1 - x_2 - x_3 \\
   w_2 &= -x_1 - 2x_2 + 3x_3 \\
   w_3 &= 2x_1 + x_3.
   \end{align*}
   \]

   Find the composite mapping $x \in \mathbb{R}^3 \mapsto u \in \mathbb{R}^3$.

14. Find the inverse of each of the following matrices:
   
   \[
   \begin{align*}
   (a) \quad & A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}; \\
   (b) \quad & B = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 4 & -2 \\ 5 & 1 & -1 \end{pmatrix}; \\
   (c) \quad & C = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}; \\
   (d) \quad & D = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & -1 & -2 & 3 \\ -1 & 2 & 2 & -4 \\ 0 & 1 & 2 & -5 \end{pmatrix}.
   \end{align*}
   \]

15. Let
   
   \[
   A = \begin{pmatrix} 1 & 2 & -1 & 0 & 4 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
   \]
16. Given are the matrices

\[
A = \begin{pmatrix}
-2 & 5 \\
1 & -3
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
1 & 4 \\
-2 & 9
\end{pmatrix}.
\]

Find \((AB)^{-1}\) and \(B^{-1}A^{-1}\).

17. Given are the following matrix equations:

(a) \((XA)^T = B\);
(b) \(XA = B - 2X\);
(c) \(AXB = C\);
(d) \(A(XB)^{-1} = C\);
(e) \(C^T XA + (X^T C)^T = I - 3C^T X\).

Find matrix \(X\).

18. Given is an open Input-Output-Model (Leontief model) with

\[
A = \begin{pmatrix}
0 & 0 & 0.2 & 0.1 & 0.3 \\
0 & 0 & 0.2 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0.4 & 0 & 0 & 0
\end{pmatrix}.
\]

Let \(x\) be the total vector of goods produced and \(y\) the final demand vector.

(a) Explain the economic meaning of the elements of \(A\).
(b) Find the linear mapping which maps all the vectors \(x\) into the set of all final demand vectors \(y\).
(c) Is a vector \(x\) of goods produced possible for some final demand vector \(y\)?
(d) Find the inverse mapping of (b) and interpret it economically.

19. A firm produces by means of three factors \(R_1, R_2\) and \(R_3\) five products \(P_1, P_2, \ldots, P_5\), where some of these products are also used as intermediate products. The relationships are given in the graph presented in Figure 1. The numbers besides the arrows describe how many units of \(R_i\) respectively \(P_i\) are necessary for one unit of \(P_j\). Let \(p_i\) denote the produced units (output) of \(P_i\) and \(q_i\) denote the final demand for the output of \(P_i\), \(i \in \{1, 2, \ldots, 5\}\).

(a) Find a linear mapping \(p \in \mathbb{R}_+^5 \mapsto q \in \mathbb{R}_+^5\).
(b) Find the inverse mapping.
(c) Let \(r^T = (r_1, r_2, r_3)\) be the vector which contains the required units of the factors \(R_1, R_2, R_3\). Find a linear mapping \(q \mapsto r\). Calculate \(r\) when \(q = (50, 40, 30, 20, 10)^T\).
Figure 1. Relationships between raw materials and products in Exercise 19