THE COMPLEXITY OF DISSOCIATION SET PROBLEMS IN GRAPHS

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Abstract. A subset of vertices in a graph is called a dissociation set if it induces a subgraph with a vertex degree of at most 1. The maximum dissociation set problem, i.e., the problem of finding a dissociation set of maximum size in a given graph is known to be NP-hard for bipartite graphs. We show that the maximum dissociation set problem is NP-hard for planar line graphs of planar bipartite graphs. In addition, we describe several polynomially solvable cases for the problem under consideration. One of them deals with the subclass of so-called chair-free graphs. Furthermore, the related problem of finding a maximal (by inclusion) dissociation set of minimum size in a given graph is studied, and NP-hardness results for this problem, namely for weakly chordal and bipartite graphs, are derived. Finally, we provide inapproximability results for the dissociation set problems mentioned above.
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1. Introduction

In this paper, we consider finite undirected simple graphs and use standard graph-theoretic terminologies, see for example Bondy and Murty [5]. For the concepts related to approximability, we follow Ausiello et al. [3].

Let G be a graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. For a subset of vertices $X \subseteq V(G)$, the subgraph of G induced by X is denoted by $G(X)$. As usual $N_G(x)$, or simply $N(x)$, denotes the *neighborhood* of a vertex $x \in V$, i.e., the set of all vertices that are adjacent to x in G. If $y \in N(x)$, then y is called a *neighbor* of x in G. The *degree* of x is defined as $\deg x = |N(x)|$. The maximum vertex degree of G is denoted by $\Delta(G)$. $K_{m,n}$ denotes the complete bipartite graph with partition classes of cardinalities m and n; K_n is the complete graph on n vertices; C_n and P_n are the chordless cycle and the chordless path on n vertices, respectively. The graph $K_{1,n}$ is also called a *star*, and $K_3 = C_3$ is called a *triangle*. At the same time, the star $K_{1,3}$ is known as a *claw*. $K_4 - e$ is a graph obtained from the complete graph K_4 by deleting an edge.

We denote by G^2 the *square* of graph G, i.e., the graph on $V(G)$ in which two vertices are adjacent if and only if they have a distance of at most 2 in G. For a graph G, the *line graph* $L(G)$ is defined as follows: the vertices of $L(G)$ bijectively correspond to the edges of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. A graph H is called a *line graph* if there exists a graph G such that

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$H = L(G)$. For vertex-disjoint graphs G_1 and G_2, the *disjoint union* $G_1 \cup G_2$ denotes the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. For a positive integer n, the disjoint union of n copies of a graph G is denoted by nG. For example, the graph mK_2 consists of m pairwise disjoint edges. A graph G is *weakly chordal* (also called *weakly triangulated*) if neither the graph G nor the complement $\overline{G}$ of this graph has an induced cycle on five or more vertices.

A class of graphs is called *hereditary* if every induced subgraph of a graph in this class also belongs to the class. For a set $\mathcal{H}$ of graphs, a graph G is called *$\mathcal{H}$-free* if no induced subgraph of G is isomorphic to a graph in $\mathcal{H}$. In other words, $\mathcal{H}$-free graphs constitute a hereditary class defined by $\mathcal{H}$ as the set of forbidden induced subgraphs.

For a graph G, a subset $D \subseteq V(G)$ is called a *dissociation set* if it induces a subgraph with a vertex degree of at most 1, i.e., $\Delta(G(D)) \leqslant 1$. A dissociation set D is *maximal* if no other dissociation set in G contains D. Let $DS(G)$ be the set of all maximal dissociation sets in G. Define the *minimum maximal dissociation number* as

$$diss^-(G) = \min\{|D| : D \in DS(G)\}$$

and the *maximum dissociation number* (also known as *1-dependence number* [19, 20]) as

$$diss^+(G) = \max\{|D| : D \in DS(G)\}.$$

A *maximum dissociation set* is a dissociation set that contains $diss^+(G)$ vertices. A *minimum maximal dissociation set* is a maximal dissociation set that contains $diss^-(G)$ vertices.

For example, all maximal dissociation sets (up to symmetry) for the path P_5 are shown in Fig. 1 as the sets of encircled vertices. In this case, $diss^+(P_5) = 4$ and $diss^-(P_5) = 3$.

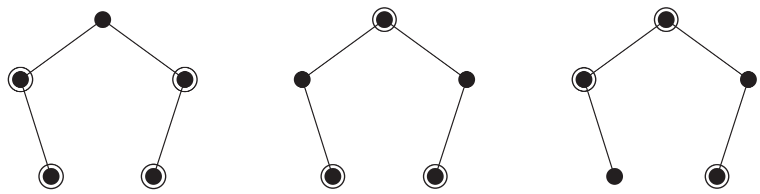


Fig. 1. Maximal dissociation sets of graph P_5.

Consider the following two decision problems associated with the parameters $diss^+(G)$ and $diss^-(G)$. We will refer to these problems as *dissociation set problems.*

MAXIMUM DISSOCIATION SET
Instance: A graph G and an integer k.
Question: Is there a dissociation set D in G such that $|D| \geqslant k$? In other words, is $diss^+(G) \geqslant k$?

This problem has been introduced by Yannakakis [58] and was shown to be NP-complete for the class of bipartite graphs. Boliac, Cameron and Lozin [6] strengthened the result of Yannakakis by showing that the problem is NP-complete for $K_{1,4}$-free bipartite graphs as well as for C_4-free bipartite graphs with a maximum vertex degree of 3. It is also known that the problem is NP-complete for planar graphs with a maximum vertex degree of 4, see Papadimitriou and Yannakakis [53]. On the other hand, the problem is polynomially solvable for chordal and weakly chordal graphs, asteroidal triple-free (AT-free) graphs [11], $(P_k, K_{1,n})$-free graphs (for any positive k and n) [42] and for some other hereditary classes of graphs [6, 11, 42].

The second problem has not been introduced before. We define it in the following way.

Minimum Maximal Dissociation Set
Instance: A graph G and an integer k.
Question: Is there a maximal dissociation set D in G such that $|D| \leqslant k$? In other words, is $diss^-(G) \leqslant k$?

The Maximum Dissociation Set problem is related to the well-known maximum independent set and maximum induced matching problems.

For a graph G, a subset $S \subseteq V(G)$ of vertices is called an *independent set* if no two vertices in S are adjacent. In other words, the degrees of all vertices of the subgraph of G induced by S are equal to 0, i.e., the subgraph $G(S)$ is 0-regular. The maximum cardinality of an independent set of G is the *independence number*, and it is denoted by $\alpha(G)$. For a graph G, a subset $M \subseteq E(G)$ of edges is called an *induced matching* if (*i*) the set M is a matching in G (a set of pairwise nonadjacent edges) and (*ii*) there is no edge in $E(G) \setminus M$ connecting two edges of M. In other words, the degrees of all vertices of the subgraph of G induced by the end-vertices of the edges of M are equal to 1, i.e., the subgraph $G(V(M))$ is 1-regular. The maximum cardinality of an induced matching of G is the *induced matching number*, and it is denoted by $\Sigma(G)$.

Consider the following two decision problems associated with the parameters $\alpha(G)$ and $\Sigma(G)$.

Maximum Independent Set
Instance: A graph G and an integer k.
Question: Is $\alpha(G) \geqslant k$?

Maximum Induced Matching
Instance: A graph G and an integer k.
Question: Is $\Sigma(G) \geqslant k$?

The optimization version of the Maximum Independent Set (the Maximum Induced Matching) problem consists in finding an independent set (an induced matching) of maximum size in a graph G.

The Maximum Independent Set problem is known to be NP-complete for general graphs [21]. Moreover, it remains NP-complete even for graphs having a specific structure, such as K_3-free graphs [50], planar graphs with a maximum vertex degree of at most 3 [21], and graphs with a large girth [45]. On the other hand, the problem can be solved in polynomial time for some hereditary classes of graphs, such as perfect graphs [25], $K_{1,3}$-free graphs [44, 46, 51], mK_2-free graphs (for any fixed $m \geqslant 2$) [1, 4], AT-free graphs [8], chair-free graphs [2], circular-arc graphs [43], and for some subclasses of P_5-free graphs [41]. Various papers such as [27, 28, 30, 59] deal with the hardness of approximating the Maximum Independent Set problem. It is known that in general graphs with n vertices, the problem cannot be approximated within a factor of $n^{1-\varepsilon}$ for any fixed $\varepsilon > 0$ unless P = NP [59], see also [30].

The Maximum Induced Matching problem is NP-complete for bipartite graphs [9, 55] and bipartite graphs with a maximum vertex degree of 3 [39], C_4-free bipartite graphs [39], line graphs [37] and for planar graphs with a maximum vertex degree of 4 [36], but on the other hand, it is polynomially solvable for chordal [9] and weakly chordal graphs [12], circular-arc graphs [24], AT-free graphs [10, 14], $(P_k, K_{1,n})$-free graphs (for any positive k and n) [42], and graphs where a maximum matching and a maximum induced matching have the same size [13, 37]. Regarding polynomial-time approximability, it is known that the Maximum Induced Matching problem is APX-complete on r-regular graphs for all $r \geqslant 3$, and bipartite graphs with a maximum vertex degree of 3 [18]. Moreover, for r-regular graphs it is NP-hard to approximate the Maximum Induced Matching problem within a factor of $r/2^{O(\sqrt{\ln r})}$ [15]. In general graphs with n vertices, the problem cannot be approximated within a factor of $n^{1/2-\varepsilon}$ for any constant $\varepsilon > 0$ unless P = NP [47].

Notice that the Maximum Dissociation Set problem asks whether in a given graph, there exists a maximum induced subgraph with any vertex degree equal to 0 or 1, while the Maximum Independent Set problem asks whether there exists a maximum induced subgraph with any vertex degree equal to 0 and the Maximum Induced Matching problem asks whether there exists a maximum induced subgraph with any vertex degree equal to 1.

Since independent sets and induced matchings are (by definition) dissociation sets, the following inequalities hold for any graph G: $\alpha(G) \leqslant diss^+(G)$ and $2\Sigma(G) \leqslant diss^+(G)$. In fact, both differences $diss^+(G) - \alpha(G)$ and $diss^+(G) - 2\Sigma(G)$ can be arbitrarily large. Indeed, for any positive integer r, let H_r be the graph formed by identifying one vertex from r copies of cycle C_7. We have $diss^+(H_r) - \alpha(H_r) = r$ since $diss^+(H_r) = 4r$ and $\alpha(H_r) = 3r$. For graph $K_{1,r+2}$, we have $diss^+(K_{1,r+2}) - 2\Sigma(K_{1,r+2}) = r$ since $diss^+(K_{1,r+2}) = r + 2$ and $\Sigma(K_{1,r+2}) = 1$.

Table 1 compiles available results on the complexity of the Maximum Dissociation Set problem (MDS), the Maximum Independent Set problem (MIS) and the Maximum Induced Matching problem (MIM) by indicating classes of graphs for which the problems are polynomially solvable (P), NP-complete (NP-c) or the complexity status of which is open (?). For the definitions of the graph classes in this table, see e.g. Brandstädt et al. [7].

Table 1. Complexity of MDS, MIM and MIS.

Graph classes / Problems	MDS	MIM	MIS
Planar graphs	NP-c [53]	NP-c [36]	NP-c [21]
Triangle-free graphs	NP-c [58]	NP-c [9, 55]	NP-c [50]
Bipartite graphs	NP-c [6, 58]	NP-c [9, 39, 55]	P [25]
Claw-free graphs	?	NP-c [37]	P [44, 46, 51]
Line graphs	?	NP-c [37]	P [44, 46, 51]
Chordal graphs	P [11]	P [9]	P [25]
Weakly chordal graphs	P [11]	P [12]	P [25]
Circular-arc graphs	P [11]	P [24]	P [43]
AT-free graphs	P [11]	P [10, 14]	P [8]

The rest of this paper is organized as follows. In Section 2, it is shown that the Maximum Dissociation Set problem is NP-complete for line graphs and therefore for claw-free graphs. In Section 3, we consider some polynomially solvable cases of the Maximum Dissociation Set problem. In Section 4, we show that the Minimum Maximal Dissociation Set problem is NP-complete for weakly chordal graphs. Finally, inapproximability results for the dissociation set problems under consideration are given in Section 5.

2. **Complexity of the maximum dissociation set problem for line graphs**

An interesting special case of the Maximum Dissociation Set problem arises when the input graph is a line graph. We show that this special case is NP-complete (Theorem 1) by a polynomial-time reduction from a variant of the following decision problem.

Partition into Isomorphic Subgraphs
Instance: Graphs G and H with $|V(G)| = q|V(H)|$ for some positive integer q.
Question: Does G have a partition into subgraphs H, i.e., is there a partition $\cup_{i=1}^{q} V_i$ of $V(G)$ such that $G(V_i)$ contains a subgraph isomorphic to H for all $i = 1, 2, \ldots, q$?

It is well-known that this problem is NP-complete for any fixed graph H that contains a connected component of three or more vertices (Kirkpatrick and Hell [34, 35], see also Garey and Johnson [21]).

Consider the special case of Partition into Isomorphic Subgraphs when H is the graph P_3: problem Partition into Subgraphs Isomorphic to P_3. Recall that P_3 is a 3-path, i.e., a graph with the vertex set $\{u, v, w\}$ and the edge set $\{uv, vw\}$.

Theorem 1. Maximum Dissociation Set *is an* NP*-complete problem for line graphs.*

Proof. Let G be a graph with $|V(G)| = 3q$ for some positive integer q. To prove the theorem, it suffices to show the following.

Claim 1. *Graph G has a partition into subgraphs P_3 if and only if the graph $H = L(G)$ has a dissociation set of a size of at least $2q$.*

Proof. It is easy and straightforward to verify that a set of q mutually vertex-disjoint 3-paths of the graph G corresponds precisely to an induced matching of size q in the line graph $H = L(G)$. Clearly, any induced matching is also a dissociation set. Thus, if G has a partition into subgraphs P_3, then H has a dissociation set of a size of at least $2q$.

Conversely, suppose that the graph $H = L(G)$ has a dissociation set $D \subseteq V(H)$ of a size of at least $2q$. Let the induced subgraph $H(D)$ consist of the disjoint union of the induced matching $M = \{e_1, e_2, \ldots, e_a\}$ of size a and the independent set $S = \{v_1, v_2, \ldots, v_b\}$ of size b, i.e., $|D| = 2a + b$. Notice that one of the sets M or S may be empty. The induced matching M in the graph H corresponds precisely to a set $M^* = \{p_1, p_2, \ldots, p_a\}$ of mutually vertex-disjoint 3-paths (not necessarily induced) in G. Let $V(M^*)$ be the vertex set of the 3-paths of M^*. Moreover, an independent set S in H corresponds precisely to a matching $S^* = \{l_1, l_2, \ldots, l_b\}$ in G. Let $V(S^*)$ be the vertex set of the edges of S^*. Since D is a dissociation set, the sets $V(M^*)$ and $V(S^*)$ are disjoint. Indeed, if any two vertices, one from the path $p_i \in M^*$ ($1 \leqslant i \leqslant a$) and the other one from the edge $l_j \in S^*$

($1 \leqslant j \leqslant b$) are identical, then an end-vertex of $e_i \in M$ would be adjacent to vertex $v_j \in S$. This contradicts the fact that e_i and v_j are a part of the dissociation set D.

Let $c = |V(G) \setminus (V(M^*) \cup V(S^*))|$. Assume on the contrary that G does not have a partition into subgraphs P_3. In this case, either $b > 0$ or $c > 0$ holds. Thus, we have

$$|V(G)| = 3a + 2b + c < 3a + 2b + c + (b + 2c) = 3(a + b + c).$$

Since $|V(G)| = 3q$, we obtain $q < a + b + c$. This, in turn, implies that

$$|D| = 2a + b = 3a + 2b + c - (a + b + c) < 3q - q = 2q,$$

i.e., $|D| < 2q$. Hence, we arrive at a contradiction to the condition that the dissociation set D has a size of at least $2q$. □

This finishes the proof of the theorem. □

The results of Orlovich et al. [47] (see Lemma 1 with Remark 2 and Theorem 2 in [47]) imply the following lemma.

Lemma 1 ([47]). Partition into Subgraphs Isomorphic to P_3 *is an* NP-*complete problem for planar bipartite graphs of a maximum vertex degree of* 4 *in which every vertex of degree* 4 *is a cut-vertex.*

Sedláček [54] proved that the line graph $L(G)$ of a planar graph G is planar if and only if the maximum vertex degree of G is at most 4 and every vertex of degree 4 is a cut-vertex. Thus, combining the proof of Theorem 1 with Lemma 1, we immediately obtain the following result.

Theorem 2. Maximum Dissociation Set *is* NP-*complete for planar line graphs of planar bipartite graphs with a maximum vertex degree of* 4.

Obviously, Theorem 2 holds for the class of line graphs of bipartite graphs. This class can be characterized in terms of forbidden induced subgraphs: a graph G is the line graph of a bipartite graph if and only if G does not contain $K_{1,3}$, $K_4 - e$ and C_{2n+1} ($n \geqslant 2$) as induced subgraphs, see Harary and Holzmann [29]. Thus, Theorem 2 shows that the Maximum Dissociation Set problem is NP-complete for $(K_{1,3}, K_4 - e, C_{2n+1} : n \geqslant 2)$-free graphs.

Corollary 1. Maximum Dissociation Set *is* NP-*complete for* $(K_{1,3}, K_4 - e, C_{2n+1} : n \geqslant 2)$*-free graphs.*

3. **Some polynomially solvable cases**

In this section, we describe new polynomially solvable cases of the Maximum Dissociation Set problem. The first of them deals with the subclass of chair-free graphs, namely (G_1, G_2, G_3)-free graphs (see Fig. 2). A *chair* is the graph consisting of the vertices a, b_1, b_2, b_3, c and the edges ab_1, ab_2, ab_3, cb_3 (see the graph G_1 in Fig. 2).

Remind that the Maximum Weight Independent Set problem (in optimization form) is the following. Given a graph G and a nonnegative weight function w on $V(G)$, determine an independent set of G having a maximum weight (where the *weight* of an independent set S is given by the sum of the weights $w(v)$ of each $v \in S$). Let $\alpha_w(G)$ denote the weight of a maximum weight independent set of G.

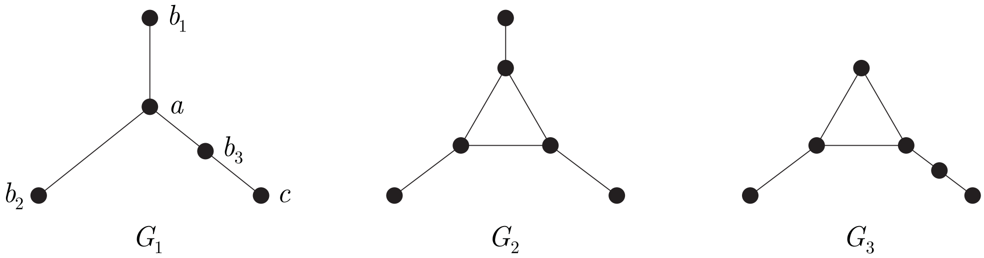


Fig. 2. Graphs G_1 (chair), G_2 and G_3.

Consider the following construction due to Lozin and Rautenbach [42]. For a graph G, let G^* denote the graph with the vertex set $V(G^*) = V(G) \cup E(G)$ such that two vertices $u, v \in V(G^*)$ are adjacent in G^* if and only if either

1. $u, v \in V(G)$ and $uv \in E(G)$ or
2. $u \in V(G)$, $v = xy \in E(G)$ and $N_G(u) \cap \{x, y\} \neq \emptyset$ or
3. $u = xy \in E(G)$, $v = zt \in E(G)$ and $(N_G(x) \cup N_G(y)) \cap \{z, t\} \neq \emptyset$.

An example of a graph G^* is shown in Fig. 3 for $G = P_5$. Here $V(P_5) = \{1, 2, 3, 4, 5\}$ and $E(P_5) = \{12, 23, 34, 45\}$. Notice that the subgraph of G^* induced by $V(G)$ coincides with G, while the subgraph of G^* induced by $E(G)$ coincides with $(L(G))^2$ [42].

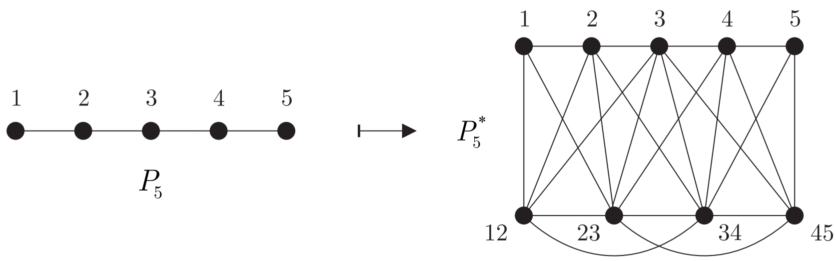


Fig. 3. Graphs P_5 and P_5^*.

Assign to each vertex v of G^* such that $v \in V(G)$ the weight $w(v) = 1$ and to each of the remaining vertices of G^* the weight 2. Lozin and Rautenbach [42] showed that the following statement holds.

Lemma 2 ([42]). *An independent set of maximum weight in G^* corresponds to a maximum dissociation set in G. In particular, $\alpha_w(G^*) = diss^+(G)$.*

For many classes $\mathcal{G}$ of graphs (e.g. chordal and weakly chordal graphs, interval-filament graphs, and AT-free graphs), it has been proved that, if a graph G is in the class $\mathcal{G}$, then the graph G^* is also in $\mathcal{G}$, see Cameron and Hell [11]. Since the time required for constructing the graph G^* is polynomial in the size of the graph G, and since finding the maximum weight independent set for the above mentioned classes can be done in polynomial time (see e.g. [8, 22, 23, 52]), one can obtain polynomial-time algorithms for the Maximum Dissociation Set problem for chordal graphs, weakly chordal graphs, interval-filament graphs, and AT-free graphs [11]. On the other hand, the class of all chair-free graphs is not closed under the transformation $G \mapsto G^*$ and therefore, we cannot apply known algorithms (see e.g. [40]) for the Maximum Weight Independent Set problem within chair-free graphs to find a maximum dissociation set in a graph from this class.

Using the construction introduced by Lozin and Rautenbach [42], we prove the following characterization theorem. In the proof of this theorem, for the sake of simplicity, a subgraph of G induced by a vertex set $\{v_1, v_2, \ldots, v_k\} \subseteq V(G)$ is denoted by $G(v_1, v_2, \ldots, v_k)$ instead of $G(\{v_1, v_2, \ldots, v_k\})$. Moreover, the notation $u \sim v$ ($u \not\sim v$, respectively) means that the vertices u and v are adjacent (nonadjacent, respectively). For disjoint sets of vertices U and W, the notation $U \sim W$ ($U \not\sim W$, respectively) means that every vertex of U is adjacent (nonadjacent, respectively) to every vertex of W. In the case when $U = \{u\}$, we also write $u \sim W$ and $u \not\sim W$ instead of $\{u\} \sim W$ and $\{u\} \not\sim W$, respectively.

Theorem 3. *The graph G^* of a graph G is* chair*-free if and only if G is (G_1, G_2, G_3)-free.*

Proof. The necessity of the condition follows from the observation that each of the graphs G_1^*, G_2^* and G_3^* contains an induced subgraph which is isomorphic to the chair.

To see the sufficiency, let G be a (G_1, G_2, G_3)-free graph. We claim that G^* is a chair-free (i.e., G_1-free) graph. Assume to the contrary that this is not the case. Then there is a set $\{a, b_1, b_2, b_3, c\} \subseteq V(G^*)$ which induces a chair G_1 in G^*, see Fig. 2. Recall that by the definition of G^*, there exists a partition $V(G^*) = V_0 \cup V_1$ such that both $G^*(V_0) = G$ and $G^*(V_1) = (L(G))^2$ hold. Furthermore, it is known that $V_0 = V(G)$ and $V_1 = E(G)$. We consider the two possible cases $a \in V_0$ and $a \in V_1$.

Case 1. $a \in V_0$. Hence, in this case, a is a vertex of the graph G. Our further discussion is split into twelve (up to symmetry) possible subcases.

Subcase 1.1. $b_1, b_2, b_3, c \in V_0$. This, however, implies that $G(a, b_1, b_2, b_3, c) = G_1$, which is a contradiction.

Subcase 1.2. $b_1, b_2, b_3 \in V_0$ and $c \in V_1$. In this case, c is an edge of the graph G, say $c = xy$. Since $\{a, c\}$ is a dissociation set in G, it follows that a, b_1, b_2 and b_3 are not the end-vertices of the edge xy. Due to the same reason and since $\{b_1, b_2, c\}$ is also a dissociation set in G, we have $\{a, b_1, b_2\} \not\sim \{x, y\}$. Notice that at least one of the vertices x, y is adjacent to b_3 in G. Without loss of generality, suppose that $x \sim b_3$. Now it is clear that $G(a, b_1, b_2, b_3, x) = G_1$, which is a contradiction.

Subcase 1.3. $b_1, b_2, c \in V_0$ and $b_3 \in V_1$. In this case, b_3 is an edge of the graph G, say $b_3 = xy$. Since $\{b_1, b_2, b_3\}$ is a dissociation set in G, it follows that the vertices a, b_1 and b_2 are not incident with xy and moreover, $\{b_1, b_2\} \not\sim \{x, y\}$. Notice that there is an edge in G connecting a and $\{x, y\}$. Without loss of generality, suppose that $a \sim x$. If $a \not\sim y$, then $G(a, b_1, b_2, x, y) = G_1$, which is a contradiction. Hence $a \sim y$ and so c is not incident with xy. Notice that vertex c is adjacent to some vertex of $\{x, y\}$ in G. Assume by symmetry that $c \sim x$. Then $G(a, b_1, b_2, x, c) = G_1$, which again gives a contradiction.

Subcase 1.4. $b_1, b_2 \in V_0$ and $b_3, c \in V_1$. In this case, both b_3 and c are edges of the graph G, say $b_3 = xy$ and $c = zt$. Notice that the vertices a, b_1 and b_2 are not incident with the edges xy and zt since $\{b_1, b_2, c\}$ and $\{b_1, b_2, b_3\}$ are dissociation sets in G. Due to the same reason and since $\{a, c\}$ is also a dissociation set in G, we have $\{b_1, b_2\} \not\sim \{x, y, z, t\}$ and $a \not\sim \{z, t\}$. Proceeding as in Subcase 1.3, we first see that $a \sim \{x, y\}$ and therefore, the edges xy and zt are not adjacent in G. Next, without loss of generality, we find that $x \sim z$. Hence $G(a, b_1, b_2, x, z) = G_1$, which is a contradiction.

Subcase 1.5. $b_1, b_3, c \in V_0$ and $b_2 \in V_1$. In this case, b_2 is an edge of the graph G, say $b_2 = xy$. It is easy to check that the vertices a, b_1, b_3 and c are not incident with xy and moreover, $\{b_1, b_3, c\} \not\sim \{x, y\}$. Notice that there is an edge in G connecting a and $\{x, y\}$. Assume by symmetry that $a \sim x$. Then $G(a, b_1, x, b_3, c) = G_1$, which is a contradiction.

Subcase 1.6. $b_1, b_3 \in V_0$ and $b_2, c \in V_1$. In this case, both b_2 and c are edges of the graph G, say $b_2 = xy$ and $c = zt$. Since $\{b_2, c\}$ is an induced matching in G, we have $\{x, y\} \not\sim \{z, t\}$. The vertices a, b_1, b_3 are not incident with the edges xy and zt since $\{b_1, b_2, b_3\}$ and $\{a, c\}$ are dissociation sets in G. Furthermore, we also have $\{b_1, b_3\} \not\sim \{x, y\}$ and $\{a, b_1\} \not\sim \{z, t\}$. Since $a \sim b_2$ ($b_3 \sim c$, respectively) in the chair G_1 (see Fig. 2), there is an edge in the graph G connecting a and $\{x, y\}$ (an edge connecting b_3 and $\{z, t\}$, respectively). Without loss of generality, let $a \sim x$ and $b_3 \sim z$. Then $G(a, b_1, x, b_3, z) = G_1$, which is a contradiction.

Subcase 1.7. $b_1, c \in V_0$ and $b_2, b_3 \in V_1$. In this case, both b_2 and b_3 are edges of the graph G, say $b_2 = xy$ and $b_3 = zt$. Since $\{b_2, b_3\}$ is an induced matching in G, $\{x, y\} \not\sim \{z, t\}$. It is easy to verify that the vertices a, b_1, c (the vertices a, b_1, respectively) cannot be incident with the edge xy (the edge zt, respectively). Moreover, $b_1 \not\sim \{x, y, z, t\}$ and $c \not\sim \{x, y\}$. Notice that vertex a is adjacent to some vertex of $\{x, y\}$ in G. Without loss of generality, suppose $a \sim x$. Arguing similarly as in the proof of Subcase 1.3, one can show that $a \sim \{z, t\}$ and therefore, c is not incident with the edge zt. Next, without loss of generality, we find that $c \sim z$. However, then $G(a, b_1, x, z, c) = G_1$, which is a contradiction.

Subcase 1.8. $b_1 \in V_0$ and $b_2, b_3, c \in V_1$. In this case, the graph G has the edges $b_2 = xy$, $b_3 = zt$ and $c = uv$. Notice that the vertices a, b_1 cannot be incident with each of the edges xy, zt or uv since $\{a, c\}$, $\{b_1, b_2\}$ and $\{b_1, b_3\}$ are dissociation sets in G. Moreover, $b_1 \not\sim \{x, y, z, t, u, v\}$ and $a \not\sim \{u, v\}$. In addition, we have $\{x, y\} \not\sim \{z, t, u, v\}$ since $\{b_2, b_3\}$ and $\{b_2, c\}$ are induced matchings in G. Since $a \sim \{b_2, b_3\}$ in the chair G_1 (see Fig. 2), there are an edge in the graph G connecting a and $\{x, y\}$ and an edge connecting a and $\{z, t\}$. Assume by symmetry that $a \sim \{x, z\}$. As in the previous subcases, one can show that $a \sim t$ and therefore, the edges zt and uv are not adjacent in G. Next, without loss of generality, we obtain that $u \sim z$. However, then $G(a, b_1, x, z, u) = G_1$, which is a contradiction.

Subcase 1.9. $b_3, c \in V_0$ and $b_1, b_2 \in V_1$. In this case, the graph G has the edges $b_1 = xy$ and $b_2 = zt$. Moreover, these edges constitute an induced matching in G. Notice that the vertices a, b_3, c cannot be incident with each of the edges xy or zt since both sets $\{b_1, b_2, b_3\}$ and $\{b_1, b_2, c\}$ are dissociation sets in G. Due to the same reason, we have $\{b_3, c\} \not\sim \{x, y, z, t\}$. Since $a \sim \{b_1, b_2\}$ in the chair G_1 (see Fig. 2), there are an edge in the graph G connecting a and $\{x, y\}$ and an edge connecting a and $\{z, t\}$. Suppose without loss of generality that $a \sim \{x, z\}$. Then $G(a, x, z, b_3, c) = G_1$, which is a contradiction.

Subcase 1.10. $b_3 \in V_0$ and $b_1, b_2, c \in V_1$. In this case, the graph G has the edges $b_1 = xy$, $b_2 = zt$ and $c = uv$, which constitute an induced matching in G. It is easy to verify that the vertices a, b_3 cannot be incident with each of the edges xy, zt or uv. Furthermore, we have $b_3 \not\sim \{x, y, z, t\}$ and $a \not\sim \{u, v\}$. Since $a \sim \{b_1, b_2\}$ and $b_3 \sim c$ in the chair G_1 (see Fig. 2), without loss of generality, we may assume that $a \sim \{x, z\}$ and $b_3 \sim u$ in G. Obviously, then $G(a, x, z, b_3, u) = G_1$, which is a contradiction.

Subcase 1.11. $c \in V_0$ and $b_1, b_2, b_3 \in V_1$. In this case, the graph G has the edges $b_1 = xy$, $b_2 = zt$ and $b_3 = uv$, which constitute an induced matching in G. It is easy to verify that the vertex a cannot be incident with each of these edges, whereas the vertex c cannot be incident with the edges xy and zt. Furthermore, we have $c \not\sim \{x, y, z, t\}$. Since $a \sim \{b_1, b_2, b_3\}$ in the chair G_1 (see Fig. 2), without loss of generality, we may assume that $a \sim \{x, z, u\}$. If $a \not\sim t$, then $G(a, x, u, z, t) = G_1$, which is a contradiction. Therefore, it follows that $a \sim t$. Similarly, one can show that $a \sim v$ and so vertex c cannot be incident with the edge uv in G. Next, without loss of generality, we obtain that $c \sim u$. However, then $G(a, x, z, u, c) = G_1$, which again gives a contradiction.

Subcase 1.12. $b_1, b_2, b_3, c \in V_1$. In this case, b_1, b_2, b_3 and c are edges in the graph G. Let $b_1 = xy$, $b_2 = zt$, $b_3 = uv$ and $c = sw$. Notice that the vertex a cannot be incident with each of these edges since $\{b_1, b_2, b_3\}$ and $\{b_1, b_2, c\}$ are induced matchings in G. Furthermore, we have $a \not\sim \{s, w\}$. Since $a \sim \{b_1, b_2, b_3\}$ in the chair G_1 (see Fig. 2), without loss of generality, we may assume that $a \sim \{x, z, u\}$. Proceeding as in Subcase 1.11, we see that $a \sim \{t, v\}$ and therefore, the edges uv and sw are not adjacent in G. Next, without loss of generality, we find that $s \sim u$. Hence $G(a, x, z, u, s) = G_1$, which is a contradiction.

Case 2. $a \in V_1$. Hence, a is an edge of the graph G. Let $V' \subseteq V(G)$ be the set of all vertices (including the end-vertices of the edges) which are contained in $\{a, b_1, b_2, b_3, c\}$. As in Case 1, there are twelve (up to symmetry) possible subcases beginning with $b_1, b_2, b_3, c \in V_0$ and finishing with $b_1, b_2, b_3, c \in V_1$. We will not present the proof of the theorem for all subcases since the basic idea of the proof is the same in each of these twelve subcases (by considering the subgraph F of G induced by V' and detecting in F one of the forbidden induced subgraphs G_1, G_2 or G_3) and due to space considerations. Thus, we restrict the proof to the following two nontrivial subcases.

Subcase 2.1. $b_1, b_2, b_3, c \in V_0$. In this case, b_1, b_2, b_3 and c are vertices in the graph G, whereas $a = xy$ is an edge in G. Since $\{a, c\}$ is a dissociation set in G, it follows that the vertices b_3, c are not incident with xy and moreover, $c \not\sim \{x, y\}$. We claim that neither b_1 nor b_2 is incident with the edge xy. Assume to the contrary that without loss of generality $b_1 = x$. Since $a \sim \{b_2, b_3\}$ in the chair G_1 (see Fig. 2), we have the only possibility that $y \sim \{b_2, b_3\}$. However, then $G(y, x, b_2, b_3, c) = G_1$, which is a contradiction. Therefore, the vertices b_1, b_2 are not incident with xy.

We denote by F the subgraph of G induced by $\{b_1, b_2, b_3, c, x, y\}$. Since $a \sim b_3$ in the chair G_1 (see Fig. 2), without loss of generality, we may assume that $y \sim b_3$. Moreover, since $a \sim \{b_1, b_2\}$ in the chair G_1, there are an edge in the graph G connecting b_1 and $\{x, y\}$ and an edge connecting b_2 and $\{x, y\}$. Note that $y \sim \{b_1, b_2\}$ leads to a contradiction since in that case, $F - x$ is isomorphic to G_1. Similarly, if $x \sim \{b_1, b_2, b_3\}$, the graph $F - y$ is isomorphic to G_1, which again gives a contradiction. Table 2 shows all remaining (up to symmetry) variants for the undefined edges b_1x, b_1y, b_2x, b_2y and b_3x.

Table 2. Remaining variants for the undefined edges in Subcase 2.1.

b_1x	b_1y	b_2x	b_2y	b_3x	Forbidden induced subgraph
No	Yes	Yes	No	No	$F - b_2$ is isomorphic to G_1
Yes	No	Yes	No	No	$F - c$ is isomorphic to G_1
Yes	No	Yes	Yes	No	F is isomorphic to G_3
No	Yes	Yes	No	Yes	F is isomorphic to G_2

Thus, we have contradictions in all cases displayed in Table 2 and therefore, Subcase 2.1 is completed.

Subcase 2.2. $b_1, b_2, b_3, c \in V_1$. In this case, the set $\{a, b_1, b_2, b_3, c\}$ induces the chair G_1 in $(L(G))^2$ since $G^*(V_1) = (L(G))^2$. In fact, Orlovich and Zverovich [49] proved that the graph $(L(G))^2$ is chair-free if and only if the graph G is $(H_1, H_2, \ldots, H_{30})$-free (see Fig. 4).

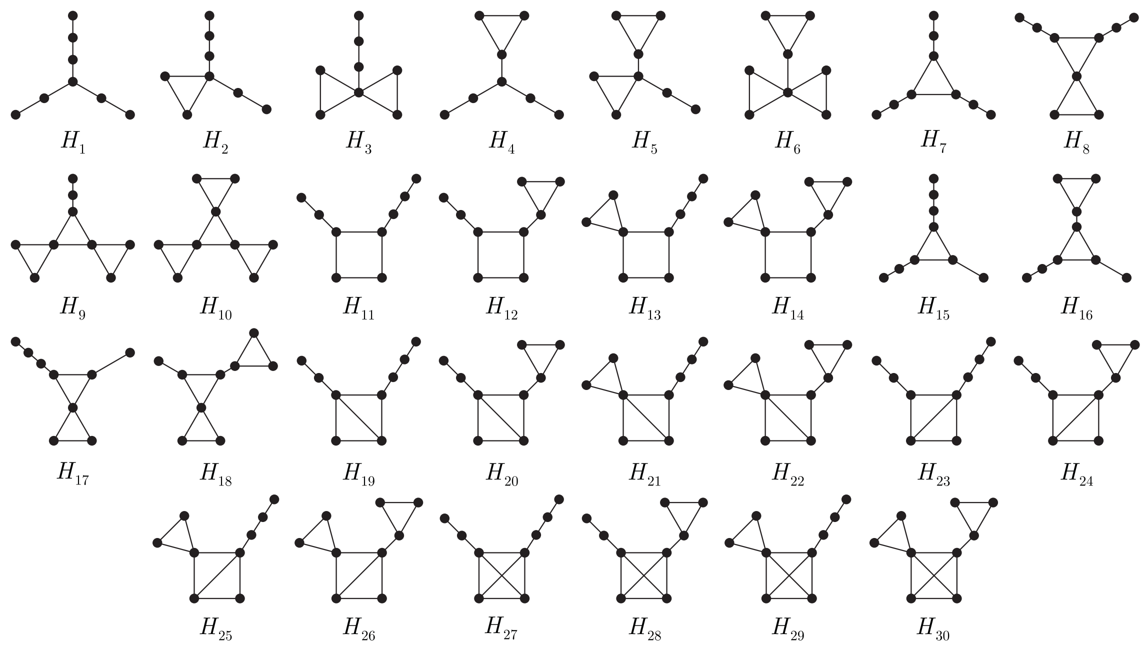


Fig. 4. Graphs $H_1 - H_{30}$.

Notice that each of the graph H_i $(1 \leqslant i \leqslant 30)$ contains at least one of the graphs G_1, G_2 or G_3 as an induced subgraph. Thus, we arrive at a contradiction to the condition that G is (G_1, G_2, G_3)-free. This completes the proof of Subcase 2.2. □

Based on the technique of modular decomposition (see e.g. [7]), the following statement has been proved by Lozin and Milanič [40].

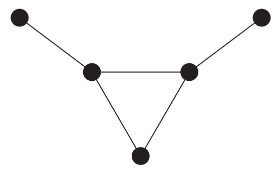
Theorem 4 ([40])**.** *The* Maximum Weight Independent Set *problem can be solved in polynomial time in the class of* chair-*free graphs.*

Lemma 2 and Theorems 3 and 4 imply the following result.

Theorem 5. *The* MAXIMUM DISSOCIATION SET *problem can be solved in polynomial time in the class of* (G_1, G_2, G_3)*-free graphs.*

Theorem 5 implies the following interesting corollary.

Corollary 2. *The* MAXIMUM DISSOCIATION SET *problem can be solved in polynomial time in the class of* (chair, bull)*-free graphs and in particular, in the class of* (claw, bull)*-free graphs, where the graph bull is shown in Fig. 5.*



bull

Fig. 5. Graph bull.

Notice that the only triangle-free graph in Fig. 2 is the chair G_1. Thus, we have the following corollary of Theorem 5.

Corollary 3. *The* MAXIMUM DISSOCIATION SET *problem can be solved in polynomial time for* (chair, K_3)*-free graphs.*

Using a similar technique as in the proof of Theorem 3, the following statement can be easily derived (we omit the proof).

Theorem 6. *Let* $m \geqslant 2$ *be an integer. The graph* G^* *of a graph* G *is* mK_2*-free if and only if* G *is* mK_2*-free.*

A general result of Balas and Yu [4] (see also Alekseev [1]) implies that the number of maximal independent sets in mK_2-free graphs (for fixed $m \geqslant 2$) is bounded by a polynomial of the graph vertex number. Using an algorithm of Tsukiyama et al. [56], these sets for a graph G can be listed in $O(nmN)$ time, where $n = |V(G)|$, $m = |E(G)|$ and N is the total number of maximal independent sets. Thus, the MAXIMUM WEIGHT INDEPENDENT SET problem for mK_2-free graphs can be solved in polynomial time by means of this algorithm. So, using Lemma 2 and Theorem 6, we have the following corollary.

Corollary 4. *For any fixed integer* $m \geqslant 2$*, the* MAXIMUM DISSOCIATION SET *problem can be solved in polynomial time in the class of* mK_2*-free graphs.*

For some classes of graphs, we can specify the complexity of finding the maximum dissociation number (Theorem 7 and Corollary 5). Remind that a simple path in a graph is called *Hamiltonian* if it contains all vertices of the graph.

Theorem 7. *Let* G *be a graph with* n *vertices and containing a Hamiltonian path. Then*

$$diss^+(L(G)) = \left\lfloor \frac{2n}{3} \right\rfloor.$$

Proof. First, we show that $diss^+(H) \geqslant \lfloor 2n/3 \rfloor$, where $H = L(G)$. As mentioned in the proof of Claim 1, a dissociation set in the graph H corresponds precisely to the disjoint union of a set M of mutually vertex-disjoint 3-paths (not necessarily induced) and a set S of mutually nonadjacent edges (i.e., a matching) in the graph G. Notice that one of the sets M or S may be empty. Grouping as far as possible successive vertices in a Hamiltonian path of the graph G into paths with three vertices and taking into account in the case of $n \equiv 2 \pmod 3$ the two remaining vertices of the Hamiltonian path, we construct the set M consisting of q mutually vertex-disjoint 3-paths and (in the case of $n \equiv 2 \pmod 3$) the set S consisting of exactly one edge, here q is the quotient when n is divided by 3. Let D be the dissociation set in H corresponding to the set $M \cup S$ in G. Then $|D| = 2q$, if $n = 3q$ or $n = 3q + 1$, and $|D| = 2q + 1$, if $n = 3q + 2$. This implies that $|D| = \lfloor 2n/3 \rfloor$. Since $diss^+(H) \geqslant |D|$, we have $diss^+(H) \geqslant \lfloor 2n/3 \rfloor$.

Conversely, let D be a maximum dissociation set in the graph H, i.e., $diss^+(H) = |D|$. As in the proof of Claim 1, suppose that the induced subgraph $H(D)$ consists of a copies of K_2 and b copies of K_1, i.e., $H(D) = aK_2 \cup bK_1$, where $a + b > 0$. Then, in the graph G, the set D corresponds precisely to the disjoint union $M^* \cup S^*$ such that M^* is the set of a mutually vertex-disjoint 3-paths and S^* is the set of b mutually nonadjacent edges. Let c denote the number of vertices in G belonging neither to the 3-paths of M^* nor to the edges of S^*. Then

$$n = 3a + 2b + c \leqslant 3a + 2b + c + (b + 2c) = 3(a + b + c),$$

which implies that $a + b + c \geqslant n/3$. Since $diss^+(H) = |D|$ and $|D| = 2a + b$, it follows by the last inequality that

$$diss^+(H) = 2a + b = 3a + 2b + c - (a + b + c) \leqslant n - \frac{n}{3} = \frac{2n}{3}.$$

Comparing the last inequality with $diss^+(H) \geqslant \lfloor 2n/3 \rfloor$, we have $diss^+(H) = \lfloor 2n/3 \rfloor$. □

By means of Theorem 7, we obtain the following corollary.

Corollary 5. *The maximum dissociation number can be computed in linear time in the class of line graphs of graphs having a Hamiltonian path.*

Proof. Let $\mathcal{H}$ be the class of line graphs of graphs having a Hamiltonian path and let $H \in \mathcal{H}$. There is an algorithm that runs in $O(|V(H)| + |E(H)|)$ time and generates a graph G such that $H = L(G)$ (see e.g. [38]). By Whitney's theorem [57], it follows that the graph G is unique if H has no component that is isomorphic to K_3. Obviously, if $H = K_3$, then $diss^+(H) = 2$. Thus, if $|V(H)| \geqslant 4$, we can find the number of vertices of the graph G in linear time. From Theorem 7, it follows that $diss^+(H) = \lfloor 2|V(G)|/3 \rfloor$, and hence, we can find the maximum dissociation number of graph H in linear time. □

Remark 1. We showed in the proof of Corollary 5 that one can find in polynomial time the maximum dissociation number $diss^+(H)$ of the line graph H of a graph having a Hamiltonian path. An interesting question is whether an algorithm with polynomial-time complexity may exist for finding a maximum dissociation set D in the graph H, i.e., a dissociation set $D \subseteq V(H)$ such that $|D| = diss^+(H)$. Notice that the previous results do not provide a natural way to ask such a question.

4. The minimum maximal dissociation set problem

In this section, we show that the Minimum Maximal Dissociation Set problem is NP-complete for weakly chordal graphs. The class of weakly chordal graphs, introduced in [31], is a well-studied class of perfect graphs, see e.g. [32].

For the proof of NP-completeness, we will use a polynomial-time reduction from the well-known NP-complete problem 3-Satisfiability, abbreviated as 3-Sat (Cook [16], see also Garey and Johnson [21]).

3-Sat
Instance: A collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over a set $X = \{x_1, x_2, \ldots, x_n\}$ of $0-1$ variables such that $|c_j| = 3$ for $j = 1, 2, \ldots, m$.
Question: Is there a truth assignment for X that satisfies all the clauses in C?

Theorem 8. Minimum Maximal Dissociation Set *is* NP-*complete for weakly chordal graphs.*

Proof. Clearly, the problem is in NP. To show that it is NP-complete, we establish a polynomial-time reduction from 3-Sat. Let $C = \{c_1, c_2, \ldots, c_m\}$ and $X = \{x_1, x_2, \ldots, x_n\}$ be an instance of 3-Sat. We construct a graph $G = G_{(C,X)}$ in the following way:

- For each variable x_i, we construct a graph F_i as follows. First, take an edge $x_i\overline{x}_i$, where the end-vertices x_i and $\overline{x}_i$ of the edge are called the *literal* vertices. Then add a triangle $(y_i, \overline{y}_i, z_i, y_i)$ and join x_i to y_i and $\overline{x}_i$ to $\overline{y}_i$, respectively. Thus, F_i is isomorphic to the graph $\overline{P}_5$, where $\overline{P}_5$ is the complement of the path P_5.
- For each clause c_j, we construct a graph consisting of one vertex c_j, where c_j is called the *clause* vertex. The set $C' = \{c_1, c_2, \ldots, c_m\}$ of all clause vertices induces a complete subgraph in G.
- For each clause $c_j = (l_j^1 \vee l_j^2 \vee l_j^3)$, introduce the three edges $c_jl_j^1$, $c_jl_j^2$ and $c_jl_j^3$ between the clause vertex c_j and the corresponding three literal vertices l_j^1, l_j^2 and l_j^3 from the set $X' = \cup_{i=1}^n V(F_i)$.

The graph G associated with the instance (C, X) of 3-Sat, where $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $C = \{c_1 = (x_1 \vee x_2 \vee x_3), c_2 = (\overline{x}_2 \vee \overline{x}_3 \vee x_4), c_3 = (\overline{x}_1 \vee x_3 \vee x_5), c_4 = (\overline{x}_3 \vee \overline{x}_4 \vee x_5)\}$ is shown in Fig. 6.

Claim 2. *The graph $G = G_{(C,X)}$ is weakly chordal.*

Proof. To prove that the graph G is weakly chordal, we show that neither the graph G nor the complement $\overline{G}$ of this graph has an induced cycle on five or more vertices.

First we show that the graph G is $(C_k : k \geqslant 5)$-free. Assume that we have an induced cycle C_k $(k \geqslant 5)$ in G. We cannot have $V(C_k) \subseteq X'$ since X' induces n disjoint copies of $\overline{P}_5$ and the graph $\overline{P}_5$ has no induced cycle on five or more vertices. Furthermore, we cannot have $V(C_k) \subseteq C'$ since C' induces a complete subgraph in G. It follows that $V(C_k) \cap C' \neq \emptyset$ and $V(C_k) \cap X' \neq \emptyset$. We must have $|V(C_k) \cap C'| \leqslant 2$ (otherwise, we have a chord).

Let $|V(C_k) \cap C'| = 2$ and $V(C_k) \cap C' = \{c_p, c_q\}$. The cycle C_k consists of two vertex-disjoint chordless paths between c_p and c_q one of which is the edge c_pc_q. The second path

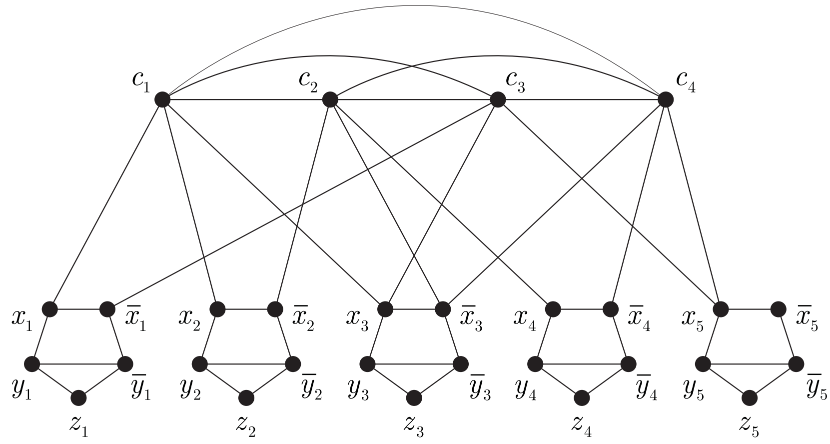


Fig. 6. An illustration of the construction.

cannot include the vertices simultaneously from $V(F_i)$ and $V(F_j)$, $i \neq j$, since there are no edges in the graph G between $V(F_i)$ and $V(F_j)$. Thus, $V(C_k) \setminus \{c_p, c_q\} \subseteq V(F_i)$ for some i, $1 \leqslant i \leqslant n$, and this gives either the chord $x_i\overline{x}_i$ or the chord $y_i\overline{y}_i$ in C_k, which is impossible.

Let $|V(C_k) \cap C'| = 1$ and $V(C_k) \cap C' = \{c_p\}$. Similarly, the cycle C_k cannot include the vertices simultaneously from $V(F_i)$ and $V(F_j)$, $i \neq j$. Thus, $V(C_k) \setminus \{c_p\} \subseteq V(F_i)$ for some i, $1 \leqslant i \leqslant n$, and this gives either the chord $x_i\overline{x}_i$ or the chord $y_i\overline{y}_i$ in C_k, which is impossible. Hence, graph G is $(C_k : k \geqslant 5)$-free.

Now we show that graph $\overline{G}$ is $(C_k : k \geqslant 5)$-free. In $\overline{G}$, the set C' is an independent set, each set $V(F_i)$, $i = 1, 2, \ldots, n$, induces a path $P_5 = (y_i, \overline{x}_i, z_i, x_i, \overline{y}_i)$ and for $i \neq j$, each vertex from $V(F_i)$ is adjacent to every vertex of $V(F_j)$. Each vertex in C' is necessarily adjacent to the vertices y_i, $\overline{y}_i$, z_i and can be adjacent to the vertices x_i, $\overline{x}_i$ of each path $P_5 = (y_i, \overline{x}_i, z_i, x_i, \overline{y}_i)$.

Assume that there is an induced cycle C_k (with $k \geqslant 5$) in $\overline{G}$. Notice that this cycle cannot include the vertex $c_p \in C'$ and the two edges c_pu and c_pv, where $u \in V(F_i)$, $v \in V(F_j)$ and $i \neq j$, since it creates the chord uv.

First, we show that the cycle C_k cannot use any two vertices c_p and c_q from C'. Assume to the contrary that $\{c_p, c_q\} \subset V(C_k)$. Let the cycle C_k include the edges c_pu, c_pv, c_qu', c_qv', where $u, v \in V(F_i)$ and $u', v' \in V(F_j)$.

If $i \neq j$, then $u = x_i$, $v = \overline{x}_i$, $u' = x_j$ and $v' = \overline{x}_j$ since otherwise, without loss of generality, we would have $u \in \{y_i, \overline{y}_i, z_i\}$, and we have the chord c_qu since c_q is adjacent to u. Since x_i is adjacent both to x_j and $\overline{x}_j$, then either uu' or uv' is a chord in C_k.

If $i = j$, then $u \neq v$, $u' \neq v'$ but it can be that $\{u, v\} \cap \{u', v'\} \neq \emptyset$. If the cycle C_k includes the edge c_pu and vertex u is adjacent to c_q in $\overline{G}$, then the edge c_qu belongs to C_k since otherwise c_qu is a chord in C_k. Thus, $\{u, v\} \nsubseteq \{y_i, \overline{y}_i, z_i\}$ and $\{u', v'\} \nsubseteq \{y_i, \overline{y}_i, z_i\}$. Let $u \in \{y_i, \overline{y}_i, z_i\}$, then one of the vertices u', v' coincides with u, say vertex u', and $v, v' \in \{x_i, \overline{x}_i\}$. If $v = v'$, then $|V(C_k)| = 4$, which is a contradiction. If $v \neq v'$, then we have either the chord uv or the chord uv' in C_k. It remains to consider the case when $u, v, u', v' \in \{x_i, \overline{x}_i\}$ which also implies a contradiction since $|V(C_k)| = 4$.

Now we show that the cycle C_k cannot use exactly one vertex c_p from C'. Assume to the contrary that $c_p \in V(C_k)$. Let the cycle C_k include the edges $c_p u$ and $c_p v$, then $u, v \in V(F_i)$ for some i, $1 \leqslant i \leqslant n$. The cycle C_k cannot include any vertex from $V(F_j)$, $j \neq i$, since otherwise $|V(C_k)| = 4$ or we have a chord in C_k. Thus, $V(C_k) \subseteq \{c_p\} \cup V(F_i)$ and we arrive at a contradiction since any cycle of a length of at least 5 in the induced subgraph $\overline{G}(\{c_p\} \cup V(F_i))$ has a chord.

We have shown that the cycle C_k (with $k \geqslant 5$) does not include any vertex from C' and therefore, $V(C_k) \subseteq X'$. Since $\overline{G}(V(F_i)) = P_5$, $1 \leqslant i \leqslant n$, and for $i \neq j$, each vertex from $V(F_i)$ is adjacent to every vertex of $V(F_j)$, the cycles in $G(X')$ without chords have four vertices. This completes the proof of the claim. □

It is easy to see that the graph G can be constructed in polynomial time in $m = |C|$ and $n = |X|$. To complete the proof, it now suffices to show the following.

Claim 3. *There exists a satisfying truth assignment for C if and only if G has a maximal dissociation set of size $2n$.*

Proof. First, suppose that there exists a truth assignment ϕ satisfying C. We construct a dissociation set D in G as follows. If $\phi(x_i) = 1$, then include the vertices x_i and y_i into D; otherwise, the vertices $\overline{x}_i$ and $\overline{y}_i$ are included into D. Clearly, D is a maximal dissociation set of size $2n$ in G.

Conversely, suppose that D is a maximal dissociation set with $|D| = 2n$. Note that any maximal dissociation set in G contains at least two vertices of each graph F_i, $i = 1, 2, \ldots, n$. Indeed, if this maximal dissociation set does not contain the vertices x_i and $\overline{x}_i$ of F_i, then it contains two vertices of the set $\{y_i, \overline{y}_i, z_i\} \subset V(F_i)$. On the other hand, if it contains at least one of the vertices x_i or $\overline{x}_i$ of F_i, then it contains z_i. Since $|D| = 2n$, the maximal dissociation set D contains exactly two vertices of each graph F_i, $i = 1, 2, \ldots, n$, and it does not contain any c_j, $j = 1, 2, \ldots, m$. Specifically, we can have one of the following three variants: $D \cap V(F_i) = \{x_i, y_i\}$, $D \cap V(F_i) = \{\overline{x}_i, \overline{y}_i\}$ or $D \cap V(F_i) = \{y_i, \overline{y}_i\}$.

Consider a clause vertex c_j and the corresponding three literal vertices l_j^1, l_j^2 and l_j^3 from $\cup_{i=1}^n V(F_i)$. Let $l_j^1 \in V(F_p)$, $l_j^2 \in V(F_q)$ and $l_j^3 \in V(F_r)$, where $1 \leqslant p, q, r \leqslant n$ and $p \neq q \neq r \neq p$. It cannot be that $D \cap V(F_i) = \{y_i, \overline{y}_i\}$ for each $i = p, q, r$ simultaneously. Otherwise, D is not maximal since we can include c_j into D. Thus, for at least one $i \in \{p, q, r\}$, we have $D \cap V(F_i) = \{x_i, y_i\}$ or $D \cap V(F_i) = \{\overline{x}_i, \overline{y}_i\}$. It follows that we can construct a satisfying truth assignment ϕ to C by setting $\phi(x_i) = 1$ if $D \cap V(F_i) = \{x_i, y_i\}$, and $\phi(x_i) = 0$ otherwise. □

This completes the proof of the theorem. □

Since the graphs $G_{(C,X)}$ appearing in the proof of Theorem 8 are clearly $(K_{1,5}, 2P_5)$-free, we obtain the following corollary.

Corollary 6. Minimum Maximal Dissociation Set *is* NP*-complete for $(K_{1,5}, 2P_5)$-free weakly chordal graphs.*

5. Inapproximability of dissociation set problems

The optimization version of the MAXIMUM DISSOCIATION SET (the MINIMUM MAXIMAL DISSOCIATION SET, respectively) problem asks for a maximum (minimum maximal, respectively) dissociation set in a graph G. In the following, we use the notations MAXIMUM DISSOCIATION SET and MINIMUM MAXIMAL DISSOCIATION SET when we refer also to the optimization versions of the problems.

In this section, we show that dissociation set problems are hard to approximate: for MINIMUM MAXIMAL DISSOCIATION SET within the class of bipartite graphs and for MAXIMUM DISSOCIATION SET for arbitrary graphs.

Recall that an algorithm is an $f(n)$-*approximation algorithm* for a minimization (maximization, respectively) problem if for each instance x of a problem of size n, it returns a solution y with a value $m(x, y)$ such that $m(x, y)/\text{opt}(x) \leqslant f(n)$ ($\text{opt}(x)/m(x, y) \leqslant f(n)$, respectively), where $\text{opt}(x)$ is the value of the optimum solution of x. An algorithm is a *constant approximation algorithm* if $f(n)$ is a constant. If an NP-optimization problem (i.e., its decision version is in NP) admits a polynomial-time $f(n)$-approximation algorithm, we say that it is *approximable within a factor of* $f(n)$.

To prove the hardness of an approximation for a given NP-optimization problem, the most common approach is to establish a gap-preserving reduction from a problem known to be NP-hard (or hard to approximate) to the problem under consideration (for more details, see e.g. [3]). Often, for proving inapproximability results, a technique for a duplication of graph vertices is used (see e.g. [17, 26, 33, 47, 48]). We introduce the following graph transformation (transformation of a graph G with a fixed vertex v into a graph F_v by the duplication of vertex v). Given a graph G, let O_p be an edgeless graph with p vertices, $p \geqslant 1$, such that $V(G) \cap V(O_p) = \emptyset$. For a fixed vertex $v \in V(G)$, define a graph F_v as follows. Let $V(F_v) = (V(G) \cup V(O_p)) \setminus \{v\}$. The vertices x and y are adjacent in F_v if and only if (i) the vertices x and y are adjacent in G, (ii) the vertices x and v are adjacent in G and $y \in V(O_p)$. We say that the graph F_v is obtained from graph G as a result of a p-*duplication of vertex* v. Notice that F_v can also be considered as the graph obtained from graph G by adding $p - 1$ new vertices which are adjacent to the vertices of the set $N_G(v)$ and which are not adjacent to each other.

First, we prove the following fact.

Lemma 3. *For each instance* (C, X) *of* 3-SAT *with a set* C *of* m *clauses and a set* X *of* n *variables and for each positive integer* t*, there exists a bipartite graph* G *on* $3n + 2tn(n + m)$ *vertices such that the following property holds for the minimum maximal dissociation number:*

$$diss^-(G) \begin{cases} \leqslant 2n, & \text{if } C \text{ is satisfiable,} \\ > 2nt, & \text{if } C \text{ is not satisfiable.} \end{cases}$$

Proof. Let $C = \{c_1, c_2, \ldots, c_m\}$ and $X = \{x_1, x_2, \ldots, x_n\}$ be an instance of 3-SAT. Consider a graph $H_{(C,X)}$ which is constructed in the following way:

- For each variable x_i, we construct a graph H_i as follows. First, take the path $(x_i, y_i, \overline{x}_i)$ on three vertices, where the end-vertices x_i and $\overline{x}_i$ of the path are the *literal* vertices. Then add the vertex z_i and join y_i to z_i. Thus, H_i is isomorphic to the graph $K_{1,3}$.

- For each clause c_j, we construct a graph consisting of one vertex c_j, where c_j is the *clause* vertex. The set $\{c_1, c_2, \ldots, c_m\}$ of all clause vertices is an independent set in $H_{(C,X)}$.
- For each clause $c_j = (l_j^1 \vee l_j^2 \vee l_j^3)$, introduce the three edges $c_j l_j^1$, $c_j l_j^2$ and $c_j l_j^3$ between the clause vertex c_j and the corresponding three literal vertices l_j^1, l_j^2 and l_j^3 from $\cup_{i=1}^n V(H_i)$.

Let $t \geqslant 1$ be an integer. We construct a graph $G = H_{(C,X),t}$ obtained from graph $H_{(C,X)}$ by a $2nt$-duplication of each clause vertex c_j, $j = 1, 2, \ldots, m$, and each vertex $z_i \in H_i$, $i = 1, 2, \ldots, n$. As a result of this $2nt$-duplication of the vertices c_j and z_i, we obtain the vertices $c_{j,k}$ and $z_{i,k}$, $k = 1, 2, \ldots, 2nt$, respectively. The graph G has the vertex set $C' \cup X' \cup Y \cup Z$, where $C' = \{c_{j,k} : j = 1, 2, \ldots, m,\ k = 1, 2, \ldots, 2nt\}$, $X' = \{x_i, \overline{x}_i : i = 1, 2, \ldots, n\}$, $Y = \{y_i : i = 1, 2, \ldots, n\}$ and $Z = \{z_{i,k} : i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, 2nt\}$ are disjoint sets. Thus, the graph G has $3n + 2tn(n + m)$ vertices and this graph is bipartite with the parts $C' \cup Y$ and $X' \cup Z$.

The graph $G = H_{(C,X),t}$ associated with an instance (C, X) of 3-Sat, where $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $C = \{c_1 = (x_1 \vee x_2 \vee x_3), c_2 = (\overline{x}_2 \vee \overline{x}_3 \vee x_4), c_3 = (\overline{x}_1 \vee x_3 \vee x_5), c_4 = (\overline{x}_3 \vee \overline{x}_4 \vee x_5)\}$, and $t = 1$ is shown in Fig. 7.

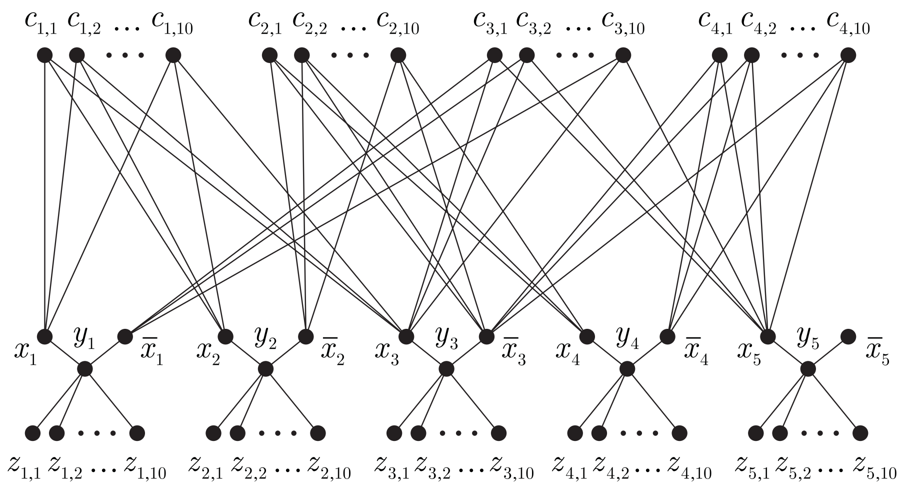


Fig. 7. An illustration of the construction.

The following statement holds.

Claim 4. *If C is satisfiable, then $diss^-(G) \leqslant 2n$ for $G = H_{(C,X),t}$, otherwise $diss^-(G) > 2nt$.*

Proof. Assume that there exists a truth assignment ϕ satisfying C. We construct a dissociation set $D \subset X' \cup Y$ by choosing the n vertices from X' that correspond to true literals under ϕ and all vertices of the set Y. That is, if $\phi(x_i) = 1$, the vertices x_i and y_i are included in D, otherwise the vertices $\overline{x}_i$ and y_i are in D.

Notice that the degree of each vertex in $G(D)$ is equal to 1. Obviously, any vertex from $(X' \setminus D) \cup Z$ has a neighbor in D. Since ϕ satisfies all the clauses in C, every vertex in C' has a neighbor in D. Thus, D is a maximal dissociation set in G. Since $|D| = 2n$ and $diss^-(G) \leqslant |D|$, we have $diss^-(G) \leqslant 2n$.

On the other hand, suppose that C is not satisfiable. Consider any maximal dissociation set D in G. It is sufficient to consider the case when $D \cap (X' \cup Y) \neq \emptyset$, since otherwise $D = C' \cup Z$ by maximality of D and we have $|D| = 2nt(m + n) > 2nt$.

Let $D \cap (X' \cup Y) \neq \emptyset$. If $D \cap X' = \emptyset$ and $D \cap Y \neq \emptyset$, then $C' \subset D$ by the maximality of D and we have $|D| > |C'| = 2mnt \geqslant 2nt$. If $D \cap X' \neq \emptyset$ and $D \cap Y = \emptyset$, then $Z \subset D$ by the maximality of D and we have $|D| > |Z| = 2n^2t \geqslant 2nt$. It remains to consider the case when $D \cap X' \neq \emptyset$ and $D \cap Y \neq \emptyset$. If $y_i \notin D$ for some $i \in \{1, 2, \ldots, n\}$, then $\{z_{i,k} : k = 1, 2, \ldots, 2nt\} \subset D$ by the maximality of D and we have $|D| > 2nt$. It remains to consider the case when $y_i \in D$ for each i, $i = 1, 2, \ldots, n$. Then for every i, $i = 1, 2, \ldots, n$, we have only one of the three possibilities: $\{x_i, y_i\} \subset D$, or $\{\overline{x}_i, y_i\} \subset D$, or $\{y_i, z_{i,k}\} \subset D$ for some $k \in \{1, 2, \ldots, 2nt\}$.

Remind that C admits no truth assignment. This means that whatever assignment we choose (i.e., whatever choice of 0 or 1 we make for each variable x_i), there will be at least one clause c_j unsatisfied (i.e., the vertices $c_{j,1}, c_{j,2}, \ldots, c_{j,2nt}$ will be not adjacent to the vertices from $D \cap X'$. Indeed, if for each $j = 1, 2, \ldots, m$, there exists a vertex in $D \cap X'$ which is adjacent to all vertices $c_{j,1}, c_{j,2}, \ldots, c_{j,2nt}$, then we construct a truth assignment ϕ by setting $\phi(x_i) = 1$ if $x_i \in D$, and $\phi(x_i) = 0$ otherwise. By the maximality of D and due to $y_i \in D$, $i = 1, 2, \ldots, n$, we can introduce the vertices $c_{j,1}, c_{j,2}, \ldots, c_{j,2nt}$ into D. Thus, $|D| > 2nt$. □

This completes the proof of the lemma.

□

Now we can present the following theorem.

Theorem 9. *Assuming that* P $\neq$ NP*,* Minimum Maximal Dissociation Set *for bipartite graphs cannot be approximated in polynomial time within a factor of* $p^{1-\varepsilon}$ *for any constant* $\varepsilon > 0$*, where* p *denotes the number of vertices in the input graph.*

Proof. For a constant $\varepsilon > 0$, we define s as follows: $s = \max\{2, \lceil 3/\varepsilon \rceil\}$. Given an instance (C, X) of 3-Sat with $|C| = m$ and $|X| = n$, we set $t = n^{s-2}$. Now we construct the bipartite graph $G = H_{(C,X),t}$ as in the proof of Lemma 3 and prove the following claim.

Claim 5. *Approximating* $diss^-(G)$ *for* $G = H_{(C,X),t}$ *within a factor of* n^{s-2} *is* NP*-hard.*

Proof. By contradiction suppose that there exists a polynomial-time n^{s-2}-approximation algorithm for the optimization version of Minimum Maximal Dissociation Set within the class of graphs $G = H_{(C,X),t}$. Then we can use this algorithm to solve 3-Sat in polynomial time which gives a contradiction to P $\neq$ NP. Indeed, applying the algorithm to G generates a dissociation set D. If C is satisfiable, then $diss^-(G) \leqslant 2n$ by Lemma 3, and therefore, $|D| \leqslant n^{s-2} diss^-(G) \leqslant 2n^{s-1}$. If C is not satisfiable, then $diss^-(G) > 2nt$ by Lemma 3 and therefore, $|D| \geqslant diss^-(G) > 2n^{s-1}$ by the choice of t. Thus, by comparing

$2n^{s-1}$ with the size of the dissociation set found by the algorithm, we solve the satisfiability of C in polynomial time. □

Now we estimate $t = n^{s-2}$ in terms of $p = |V(G)| = 3n + 2n^{s-1}(n+m)$. For this purpose, we may assume that $n \geqslant 5$ and $n = m$. Obviously, 3-SAT remains NP-complete under these additional restrictions. Indeed, if $n < m$, we can add $m - n$ dummy variables which do not occur in any clause, and if $m < n$, we can add $n - m$ trivially satisfiable clauses. Using the assumption $n = m$, we have $p > n^s$ and

$$n^{s-2} = \frac{p-3n}{2n(n+m)} = \frac{p-3n}{4n^2} > \frac{p-3n}{4p^{2/s}}.$$

Since $p = 3n + 4n^s$, $s \geqslant 2$ and $n \geqslant 5$, we have $p > 15n$ and therefore, $p - 3n > \frac{4}{5}p$. Dividing both sides of the last inequality by $4p^{2/s}$, we have

$$\frac{p-3n}{4p^{2/s}} > \frac{1}{5}p^{1-2/s}.$$

From $p = 3n+4n^s$ and $n \geqslant 5$, we have $p > 5^s$ and therefore, $p^{1/s} > 5$. Taking into account that $1/s = (1-2/s) - (1-3/s)$, we obtain

$$\frac{1}{5}p^{1-2/s} > p^{1-3/s}$$

and hence

$$t = n^{s-2} > p^{1-3/s}.$$

Since $p^{1-3/s} \geqslant p^{1-\varepsilon}$ by the definition of s, approximating $diss^{-}(G)$ within a factor of $p^{1-\varepsilon}$ is NP-hard according to Claim 5. The proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 9.

Corollary 7. MINIMUM MAXIMAL DISSOCIATION SET *is NP-complete for bipartite graphs.*

Now we show that the MAXIMUM DISSOCIATION SET problem is hard to approximate for arbitrary graphs by a reduction from the MAXIMUM INDEPENDENT SET problem.

Håstad [30] proved that MAXIMUM INDEPENDENT SET cannot be approximated in polynomial time within a factor of $|V(G)|^{1-\varepsilon}$ for each constant $\varepsilon > 0$ unless NP = ZPP. Here, ZPP denotes the class of languages decidable by a random expected polynomial-time algorithm that makes no errors. In view of the recent paper by Zuckerman [59], who derandomized Håstad's randomized reduction, "unless NP = ZPP" in the above inapproximability result for MAXIMUM INDEPENDENT SET can be changed to "unless P = NP", and we can pass to the following theorem.

Theorem 10. *Assuming that* P ≠ NP*,* MAXIMUM DISSOCIATION SET *cannot be approximated in polynomial time within a factor of* $p^{1/2-\varepsilon}$ *for any constant* $\varepsilon > 0$*, where* p *is the number of vertices in the input graph.*

Proof. We construct a polynomial-time reduction from the MAXIMUM INDEPENDENT SET problem for arbitrary graphs. Given a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, we construct a new graph H by a 2-duplication of each vertex v_i, $i = 1, 2, \ldots, n$. Each vertex v_i in G is transformed into two vertices v_i and v_i' in H.

Claim 6. *The size of a maximum dissociation set of H is equal to the double size of a maximum independent set of G, i.e., $diss^+(H) = 2\alpha(G)$.*

Proof. If S is a maximum independent set in G, then the set $D = \{v_i, v_i' : v_i \in S\}$ is an independent set in H and therefore, it is a dissociation set in H. Thus, $diss^+(H) \geqslant |D| = 2|S| = 2\alpha(G)$.

Conversely, let D be a maximum dissociation set of H, i.e., $|D| = diss^+(H)$. We can construct an independent set S in G such that $|S| = |D|/2$ in the following way. Let $u \in D$ and $\deg u = 0$ in the graph $H(D)$. If $u \in \{v_i, v_i'\}$ for some i, $1 \leqslant i \leqslant n$, then both vertices v_i and v_i' are in D due to the maximality of the dissociation set D. In this case, we include the vertex v_i into S. Let $u \in D$ with $\deg u = 1$ in the graph $H(D)$ and $w \in D$ be adjacent to u. Assume that $u \in \{v_i, v_i'\}$ and $w \in \{v_j, v_j'\}$ for some i and j, $1 \leqslant i \neq j \leqslant n$. In any case, we can suppose that either $u = v_i$ or $w = v_j$ and include either v_i or v_j into S. Indeed, if $u = v_i'$ and $w = v_j'$, we can replace the set D by the dissociation set $(D \setminus \{v_i', v_j'\}) \cup \{v_i, v_j\}$ with the same cardinality as D. Since D is a dissociation set, the constructed set S is an independent set in G and $|S| = |D|/2$. Thus, $\alpha(G) \geqslant |S| = diss^+(H)/2$, i.e., $diss^+(H) \leqslant 2\alpha(G)$.

As a result, we have $diss^+(H) = 2\alpha(G)$. □

Since $p = |V(H)| = 2|V(G)|$ and, unless P = NP, Maximum Independent Set cannot be approximated in polynomial time within a factor of $|V(G)|^{1-\varepsilon}$ for each constant $\varepsilon > 0$, we obtain that the Maximum Dissociation Set problem cannot be approximated in polynomial time within a factor of $(p/2)^{1-\varepsilon}$ and therefore, within a factor of $p^{1/2-\varepsilon}$. □

Notice that Theorems 9 and 10 give a negative answer to the question about the existence of approximation algorithms with a constant factor for the dissociation set problems.

6. **Conclusion**

In this paper, we considered the complexity of finding a dissociation set of maximum size in line graphs and finding a maximal dissociation set of minimum size. We have shown that the Maximum Dissociation Set problem is NP-complete for planar line graphs of planar bipartite graphs with a maximum vertex degree of 4. On the other hand, we have shown that the Maximum Dissociation Set problem can be solved in polynomial time for some special classes of graphs, in particular, for (G_1, G_2, G_3)-free graphs (see Fig. 2). This class includes (chair, bull)-free and (claw, bull)-free graphs as proper subclasses. Moreover, we have shown that the maximum dissociation number can be computed in linear time in the class of line graphs of graphs having a Hamiltonian path.

The Minimum Maximal Dissociation Set problem has been shown to be NP-complete for weakly chordal graphs and for bipartite graphs. For further research, it is interesting to establish the complexity of the Minimum Maximal Dissociation Set problem for such structured classes of graphs as chordal graphs, comparability graphs, circular-arc graphs, and AT-free graphs.

We have given a negative answer to the question about the existence of approximation algorithms with a constant factor for the dissociation set problems provided that P ≠ NP.

Namely, we have shown that for any constant $\varepsilon > 0$ (i) the Minimum Maximal Dissociation Set problem is hard to approximate in polynomial time within a factor of $n^{1-\varepsilon}$ even for bipartite graphs and (ii) the Maximum Dissociation Set problem is hard to approximate in polynomial time for arbitrary graphs within a factor of $n^{1/2-\varepsilon}$, where n denotes the number of vertices in the graph.

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