HAMILTONIAN PROPERTIES OF TRIANGULAR GRID GRAPHS

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ABSTRACT. A triangular grid graph is a finite induced subgraph of the infinite graph associated with the two-dimensional triangular grid. In 2000, Reay and Zamfirescu showed that all 2-connected, linearly convex triangular grid graphs (with the exception of one of them) are hamiltonian. The only exception is a graph D which is the linearly-convex hull of the Star of David. We extend this result to a wider class of locally connected triangular grid graphs. Namely, we prove that all connected, locally connected triangular grid graphs (with the same exception of graph D) are hamiltonian. Moreover, we present a sufficient condition for a connected graph to be fully cycle extendable.
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1. Introduction

In this paper, we consider hamiltonian properties of finite induced subgraphs of a graph associated with the two-dimensional triangular grid (called triangular grid graphs). Such properties are important in applications connected with problems arising in molecular biology (protein folding) [1], in configurational statistics of polymers [5, 11], in telecommunications and computer vision (problems of determining the shape of an object represented by a cluster of points on a grid). Cyclic properties of triangular grid graphs can also be used in the design of cellular networks since these networks are generally modelled as induced subgraphs of the infinite two-dimensional triangular grid [8].

For graph-theoretic terminology not defined in this paper, the reader is referred to [2]. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph is *connected* if there is a path between every pair of its vertices, and a graph is *k-connected* ($k \geqslant 2$) if there are k vertex-disjoint paths between every pair of its vertices. For each vertex u of G, the *neighborhood* $N(u)$ of u is the set of all vertices adjacent to u. The *degree* of u is defined as $\deg u = |N(u)|$. For a subset of vertices $X \subseteq V(G)$, the subgraph of G induced by X is denoted by $G(X)$. A vertex u of G is said to be *locally connected* if $G(N(u))$ is connected. G is called *locally connected* if each vertex of G is locally connected.

We say that G is *hamiltonian* if G has a *hamiltonian cycle*, i.e., a cycle containing all vertices of G. A path with the end vertices u and v is called a (u,v)*-path*. A (u,v)-path

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is a *hamiltonian path* of G if it contains all vertices of G. As usual, P_k and C_k denote the path and the cycle on k vertices, respectively. In particular, C_3 is a *triangle*. The path P (respectively, cycle C) on k vertices $v_1, v_2, \ldots, v_k$ with the edges $v_i v_{i+1}$ (respectively, $v_i v_{i+1}$ and $v_1 v_k$) ($1 \leqslant i < k$) is denoted by $P = v_1 v_2 \ldots v_k$ (respectively, $C = v_1 v_2 \ldots v_k v_1$).

A cycle C in a graph G is *extendable* if there exists a cycle C' in G (called the *extension* of C) such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. If such a cycle C' exists, we say that C *can be extended to* C'. If every non-hamiltonian cycle C in G is extendable, then G is said to be *cycle extendable*. We say that G is *fully cycle extendable* if G is cycle extendable and each of its vertices is on a triangle of G. Clearly, any fully cycle extendable graph is hamiltonian.

The infinite graph T^∞ associated with the two-dimensional triangular grid (also known as *triangular tiling graph* [7, 17]) is a graph drawn in the plane with straight-line edges and defined as follows. The vertices of T^∞ are represented by a linear combination $x\boldsymbol{p} + y\boldsymbol{q}$ of the two vectors $\boldsymbol{p} = (1, 0)$ and $\boldsymbol{q} = \big(1/2, \sqrt{3}/2\big)$ with integers x and y. Thus we may identify the vertices of T^∞ with pairs (x, y) of integers. Two vertices of T^∞ are adjacent if and only if the Euclidean distance between them is equal to 1 (see Fig. 1). Note that the degree of any vertex of T^∞ is equal to six. A *triangular grid graph* is a finite induced subgraph of T^∞. A triangular grid graph G is *linearly convex* if, for every line l which contains an edge of T^∞, the intersection of l and G is either a line segment (a path in G), or a point (a vertex in G), or empty. For example, the triangular grid graph G (with three components including an isolated vertex w) shown in Fig. 2 is linearly convex even though G has vertices u and v whose midpoint z is a vertex of T^∞ but not of G. In Fig. 2, dark points correspond to the vertices of T^∞.

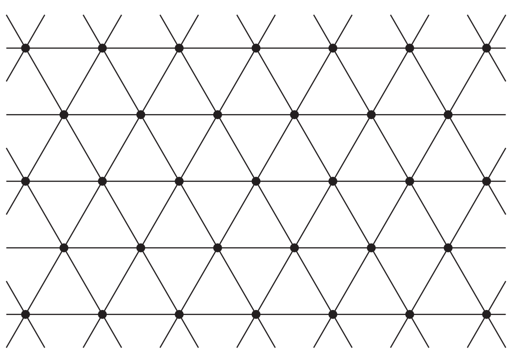

Fig. 1. A fragment of graph T^∞

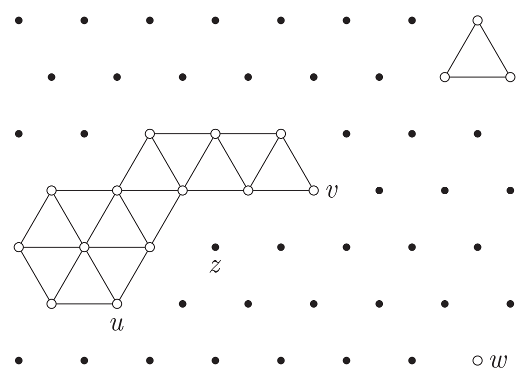

Fig. 2. A linearly convex triangular grid graph

It is well-known that the problem of deciding whether a given graph is hamiltonian, is NP-complete, and it is natural to look for conditions for the existence of a hamiltonian cycle for special classes of graphs. Our goal here is to determine such conditions for triangular grid graphs and for a wider class of graphs with the special structure of local connectivity.

The concept of local connectivity of a graph has been introduced by Chartrand and Pippert [3]. Oberly and Sumner [12] have shown that a connected, locally connected claw-free graph G on $n \geqslant 3$ vertices is hamiltonian (a graph is claw-free if it has no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$). Clark [4] has proved that, under the Oberly – Sumner's conditions, G is vertex pancyclic (i.e., every vertex of G is on cycles of length $3, 4, \ldots, n$). Later, Hendry [10] has introduced the concept of cycle extendability and strengthened Clark's result showing that, under the same conditions, G is fully cycle extendable. Some further strengthenings of these results can be found in the survey by Faudree et al. [6].

Hendry [9] has shown that connected, locally connected graphs in which the maximum and minimum degrees differ by at most one and do not exceed five are fully cycle extendable. Orlovich has improved this result by finding a stronger sufficient condition for fully cycle extendability [13] and described all connected, locally connected graphs whose maximum degree is at most four [14].

As has been shown by Reay and Zamfirescu [17], all 2-connected, linearly convex triangular grid graphs (or T-graphs in the terminology of [17]) are hamiltonian (with the exception of one of them). The only exception is a graph D which is the linearly-convex hull of the Star of David; this graph is 2-connected and linearly convex but not hamiltonian (see Fig. 3). We extend this result to a wider class of locally connected triangular grid graphs. As will be seen later, any 2-connected, linearly convex triangular grid graph is a locally connected triangular grid graph. But the converse is not true and an example can be found in Fig. 4. This example shows a connected, locally connected triangular grid graph which is not linearly convex: the intersection of the graph and the dashed line which contains edges of T^∞ is the union of a line segment (the edge vw of the graph) and a point (the vertex u of the graph).

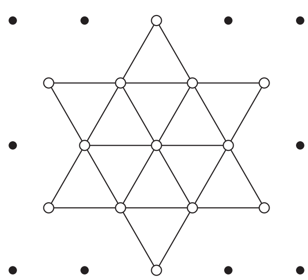

Fig. 3. Graph D

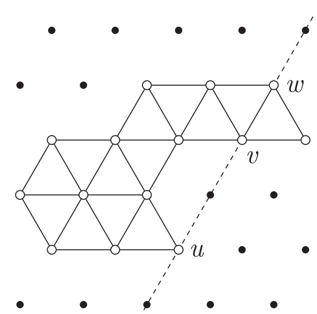

Fig. 4. A locally connected, but not linearly convex triangular grid graph

This paper is organized as follows. In Section 2, we show that any 2-connected, linearly convex triangular grid graph is locally connected (Theorem 1). In Section 3, it is proven that a connected graph of special local structure is either fully cycle extendable or isomorphic to the graph D (Theorem 2). Corollaries 1 and 2 show that this is also valid for connected, locally connected triangular grid graphs, and for 2-connected, linearly convex triangular grid graphs. The results of this paper were announced in [15, 16].

2. **Local connectivity of triangular grid graphs**

We establish an interrelation between classes of 2-connected, linearly convex triangular grid graphs and locally connected triangular grid graphs in the following theorem.

Theorem 1. *Let G be a 2-connected triangular grid graph. If G is linearly convex, then G is locally connected.*

Proof. The proof will be done by contradiction. We first introduce some useful additional notation. Recall that the vertices of T^∞ are identified with pairs (x, y) of integers. Therefore, each vertex (x, y) has six neighbors $(x \pm 1, y)$, $(x, y \pm 1)$, $(x+1, y-1)$ and $(x-1, y+1)$. For simplicity, we will refer to the neighbors of (x, y) as R (right), L (left), UR (up-right), DL (down-left), DR (down-right) and UL (up-left), respectively (see Fig. 5). For example, the notation $v = \mathrm{UR}(u)$ means that vertex v is the up-right neighbor of vertex u.

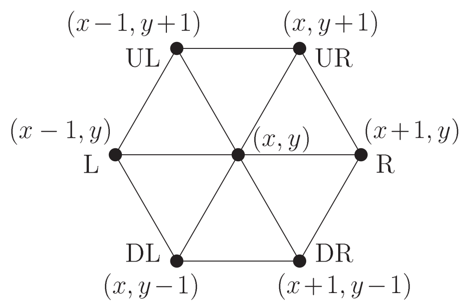

Fig. 5. Neighbors of the vertex (x, y)

Let G be a 2-connected, linearly convex triangular grid graph. Assume, to the contrary, that G contains a vertex u which is not locally connected. Note that $\deg u \leqslant 4$ (otherwise $G(N(u))$ is connected and isomorphic to P_5 if $\deg u = 5$ or to C_6 if $\deg u = 6$). On the other hand, the 2-connectedness of G implies $\deg u \geqslant 2$. Consider the three possible cases for the degree of u.

Case 1. $\deg u = 2$.

Let $N(u) = \{v, w\}$. By symmetry, we need only consider two subcases: $v = \mathrm{UR}(u)$, $w = \mathrm{DL}(u)$ (Fig. 6a), and $v = \mathrm{UR}(u)$, $w = \mathrm{DR}(u)$ (Fig. 6b). Since G is 2-connected, there exists a (v, w)-path P in G with internal vertices different from u. Let l be a line which contains the edge $u\mathrm{R}(u)$ of T^∞. Then the intersection of l and G contains vertex u as an isolated vertex (since $\mathrm{L}(u)$ and $\mathrm{R}(u)$ are not in G) and at least one vertex of P. This contradicts the condition that G is linearly convex.

Case 2. $\deg u = 3$.

Let $N(u) = \{v, w, z\}$. By symmetry, there are two subcases: $v = \mathrm{UR}(u)$, $w = \mathrm{DR}(u)$, $z = \mathrm{UL}(u)$ (Fig. 6c), and $v = \mathrm{UR}(u)$, $w = \mathrm{DR}(u)$, $z = \mathrm{L}(u)$ (Fig. 6d). In the first subcase, the proof is similar to the proof in Case 1. Consider the second subcase. Since G is 2-connected, there exists a (v, w)-path P in G with internal vertices different from u. Let l_1 be a line which contains the edge uw of T^∞, and l_2 be a line which contains the edge uz

of T^∞. Obviously, the intersection of l_1 and G contains the edge uw and does not contain $\mathrm{UL}(u)$, and the intersection of l_2 and G contains the edge uz and does not contain $\mathrm{R}(u)$. On the other hand, the intersection of these lines and graph G contains at least one vertex of path P either on the ray l' or on the ray l''. Here l' and l'' are the rays (parts of the lines l_1 and l_2) which start from u, and pass $\mathrm{UL}(u)$ and $\mathrm{R}(u)$, respectively. Hence, we arrive at a contradiction to the condition that G is linearly convex.

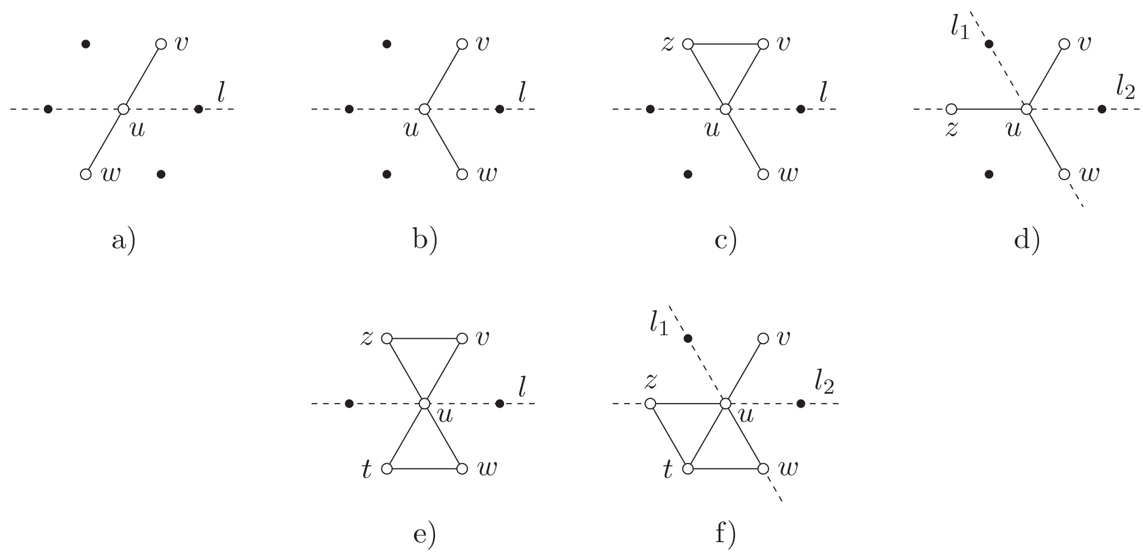

Fig. 6. Cases 1–3

Case 3. $\deg u = 4$.

Let $N(u) = \{v, w, z, t\}$. By symmetry, there are two subcases: $v = \mathrm{UR}(u)$, $w = \mathrm{DR}(u)$, $t = \mathrm{DL}(u)$, $z = \mathrm{UL}(u)$ (Fig. 6e), and $v = \mathrm{UR}(u)$, $w = \mathrm{DR}(u)$, $t = \mathrm{DL}(u)$, $z = \mathrm{L}(u)$ (Fig. 6f). The proof is similar to the proof in Case 2. □

Thus the example in Fig. 4 and Theorem 1 show that 2-connected, linearly convex triangular grid graphs form a proper subclass of the class of connected, locally connected triangular grid graphs. Note that the graphs of this class (except an isolated vertex and a complete graph on two vertices) are also 2-connected due to a well-known observation of Chartrand and Pippert [3] that a connected, locally k-connected graph is $(k+1)$-connected.

3. **Cycle extendability of locally connected triangular grid graphs**

In this section, we consider connected graphs on $n \geqslant 3$ vertices. For the sake of simplicity, a subgraph of G induced by $S = \{v_1, v_2, \ldots, v_k\} \subseteq V(G)$ is denoted by $G(v_1, v_2, \ldots, v_k)$ instead of $G(\{v_1, v_2, \ldots, v_k\})$. Before we proceed, we first state the following obvious observations.

Observation 1. *Every edge of a locally connected graph G lies in a triangle of G.*

Observation 2. *Let G be a locally connected graph and its edge uv be contained in only one triangle $G(u, v, w)$. The edge uw (the edge vw, respectively) is contained in at least two triangles of G if and only if* $\deg u \geqslant 3$ ($\deg v \geqslant 3$, *respectively*).

Observation 3. *If G is a locally connected triangular grid graph, then for any vertex u of G the subgraph $G(N(u))$ is isomorphic to one of the following five graphs: P_2, P_3, P_4, P_5, and C_6.*

The main result of this section is the following.

Theorem 2. *Let G be a connected graph such that for any vertex u of G the subgraph $G(N(u))$ is isomorphic to one of the graphs P_2, P_3, P_4, P_5, or C_6. Then G is either graph D or fully cycle extendable.*

Proof. Since for every vertex u of G the subgraph $G(N(u))$ is isomorphic to one of the graphs P_2, P_3, P_4, P_5, or C_6, graph G is locally connected. Moreover, every vertex of G has a degree of at least 2 and lies on a triangle of G. Now suppose that G is not fully cycle extendable, i.e., there exists a non-extendable, non-hamiltonian cycle $C = u_1u_2 \dots u_ku_1$ on $k < n$ vertices in G. In what follows, the subscripts of the vertices in C are taken modulo k.

Since G is connected, there exists a vertex x not on C which is adjacent to a vertex lying on C. Without loss of generality, let u_1 be a vertex on C adjacent to x. Then $3 \leqslant \deg u_1 \leqslant 6$. We proceed via the series of Claims $1 - 13$ toward a final contradiction.

First, we introduce some notation for shortening the representation of a cycle. Let the orientation of cycle $C = u_1u_2 \dots u_ku_1$ be from u_1 to u_k. For $u_i, u_j \in V(C)$, we denote by u_iCu_j the consecutive vertices on C from u_i to u_j in the direction specified by the orientation of C. The same vertices in reverse order are denoted by $u_j\overline{C}u_i$. We will consider u_iCu_j and $u_j\overline{C}u_i$ both as paths and as vertex sets. If $i = j$, then $u_iCu_j = u_j\overline{C}u_i = \{u_i\}$.

The notation $u \sim v$ ($u \not\sim v$, respectively) means that vertices u and v are adjacent (non-adjacent, respectively). For disjoint sets of vertices U and V, the notation $U \sim V$ ($U \not\sim V$, respectively) means that every vertex of U is adjacent (non-adjacent, respectively) to every vertex of V. In case of $U = \{u\}$, we also write $u \sim V$ and $u \not\sim V$ instead of $\{u\} \sim V$ and $\{u\} \not\sim V$, respectively. The notation $F \cong H$ means that graph F is isomorphic to graph H.

Define $S = V(G) \setminus V(C)$.

Claim 1. *Let $u_i \in V(C)$ and $y \in S$. If $y \sim u_i$, then $y \not\sim \{u_{i-1}, u_{i+1}\}$.*

Proof. If, to the contrary, $y \sim u_{i-1}$ or $y \sim u_{i+1}$, then C can be extended to the cycle $yu_iCu_{i-1}y$ or to the cycle $yu_{i+1}Cu_iy$, respectively. This contradicts the non-extendability of C. □

Claim 2. *Inequality* $\deg u_1 \geqslant 4$ *holds.*

Proof. Assume, to the contrary, that $\deg u_1 = 3$, i.e., $N(u_1) = \{u_2, u_k, x\}$. By Claim 1, $x \not\sim \{u_2, u_k\}$. Therefore, $G(N(u_1))$ is not isomorphic to P_3, in contradiction to $\deg u_1 = 3$. □

Claim 3. *All neighbors of u_1 except x are on C.*

Proof. Let $W = N(u_1) \setminus \{x\}$. Assume, to the contrary, that $W \cap S \neq \emptyset$. Let $y \in W \cap S$. We distinguish the three possible cases for the degree of u_1.

Case 1. Let $\deg u_1 = 4$, i.e., $N(u_1) = \{u_2, u_k, x, y\}$.

By Claim 1, $\{x, y\} \not\sim \{u_2, u_k\}$. Hence, $G(N(u_1))$ is not isomorphic to P_4, in contradiction to $\deg u_1 = 4$.

Case 2. Let $\deg u_1 = 5$ and $N(u_1) = \{u_2, u_k, x, y, z\}$.

If $z \in S$, we arrive at a contradiction by similar arguments as in Case 1. Hence $z \in V(C)$, i.e., $z = u_i$ and $3 \leqslant i \leqslant k-1$. Under the condition of this case, $G(N(u_1)) \cong P_5$. Let us show that $x \sim y$ and $u_2 \sim u_k$. By Claim 1, $\{x, y\} \not\sim \{u_2, u_k\}$. If $x \not\sim y$, we get that $u_i \sim \{x, y\}$ and u_i is adjacent to one of the vertices u_2 or u_k. Hence, $\deg u_i > 2$ in $G(N(u_1))$, in contradiction to $G(N(u_1)) \cong P_5$. The proof of $u_2 \sim u_k$ is similar.

Since $G(N(u_1)) \cong P_5$, we can assume without loss of generality that $u_i \sim \{u_2, x\}$. Therefore, $\deg u_i \geqslant 5$ in G. Note that $4 \leqslant i \leqslant k-2$, since otherwise C can be extended to the cycle $xu_3Cu_ku_2u_1x$ if $i = 3$ or to the cycle $xu_1u_ku_2Cu_{k-1}x$ if $i = k-1$. Now we can easily see that $u_{i+1} \not\sim \{u_2, u_{i-1}\}$ since otherwise C can be extended to the cycle $xu_i\overline{C}u_2u_{i+1}Cu_1x$ if $u_{i+1} \sim u_2$ and to the cycle $xu_iu_2Cu_{i-1}u_{i+1}Cu_1x$ if $u_{i+1} \sim u_{i-1}$. By Claim 1, $x \not\sim \{u_{i-1}, u_{i+1}\}$. Hence, $G(N(u_i))$ is not isomorphic to P_5 or C_6, in contradiction to $\deg u_i \geqslant 5$.

Case 3. Let $\deg u_1 = 6$ and $N(u_1) = \{u_2, u_k, x, y, z, w\}$.

Similarly to Case 2 we conclude that $z \in V(C)$ and $w \in V(C)$. Assume without loss of generality that $z = u_i$ and $w = u_j$, where $3 \leqslant i < j \leqslant k-1$. Let us show that $x \sim y$. By Claim 1, $\{x, y\} \not\sim \{u_2, u_k\}$. Under the condition of Case 3, $G(N(u_1)) \cong C_6$. Hence, the degrees of x and y in $G(N(u_1))$ are equal to 2. Therefore, if $x \not\sim y$, we have $\{x, y\} \sim \{u_i, u_j\}$ and obtain a cycle on four vertices, in contradiction to $G(N(u_1)) \cong C_6$. Thus, $x \sim y$ and, consequently, $u_i \not\sim u_j$ and $u_{j-1} \neq u_i$. By symmetry and since the degrees of x and y are equal to 2 in $G(N(u_1))$, we can assume without loss of generality that $x \sim u_j$ and $y \sim u_i$. If $u_2 \not\sim u_k$, then $\{u_2, u_k\} \sim \{u_i, u_j\}$ and we have a contradiction to $\deg u_i = 2$ in $G(N(u_1))$. Since $G(N(u_1)) \cong C_6$, there are two possibilities: $u_2 \sim u_i$, $u_j \sim u_k$ and $u_2 \sim u_j$, $u_i \sim u_k$. In the first subcase, we can obtain $4 \leqslant i \leqslant j-2$, $u_{i-1} \not\sim u_{i+1}$ and $u_2 \not\sim u_{i+1}$ similarly to the proof in Case 2 (we have only to consider vertex y instead of x). The consideration of the second subcase ($u_2 \sim u_j$, $u_i \sim u_k$) is analogous. □

Claim 4. *Relation* $\deg u_1 \neq 4$ *holds.*

Proof. Assume, to the contrary, that $\deg u_1 = 4$. Then $N(u_1) = \{u_2, u_k, x, y\}$ and, by Claim 3, $y \in V(C)$. Let $y = u_i$, $3 \leqslant i \leqslant k-1$. By Claim 1, $x \not\sim \{u_2, u_k\}$. Since $G(N(u_1)) \cong P_4$ under the assumption $\deg u_1 = 4$, we have $x \sim u_i$ and therefore $u_i \sim u_2$ or $u_i \sim u_k$. By symmetry, we can assume without loss of generality that $u_i \sim u_2$ and therefore $u_2 \sim u_k$. Similarly to the proof in Case 2 of Claim 3 we can obtain that

$4 \leqslant i \leqslant k-2$, $u_{i-1} \not\sim u_{i+1}$ and $u_2 \not\sim u_{i+1}$. Note that $u_i \sim \{u_1, u_2, u_{i-1}, u_{i+1}, x\}$ and therefore $G(N(u_i)) \cong P_5$ or $G(N(u_i)) \cong C_6$. By Claim 1, $x \not\sim \{u_{i-1}, u_{i+1}\}$. Hence, $u_{i+1} \not\sim \{u_1, u_2, u_{i-1}, x\}$ in contradiction to both $G(N(u_i)) \cong P_5$ and $G(N(u_i)) \cong C_6$. □

Claim 5. *If* $\deg x \geqslant 3$*, then the subgraphs* $G(u_1, u_3, x)$ *and* $G(u_1, u_{k-1}, x)$ *are triangles in* G*.*

Proof. Notice that, by Claims 2 and 4, $\deg u_1 \in \{5, 6\}$. Moreover, if $\deg u_1 = 6$, then $G(N(u_1)) \cong C_6$ and hence, $\deg x \geqslant 3$. Therefore, the condition of Claim 5 (i.e., $\deg x \geqslant 3$) is essential only if $\deg u_1 = 5$.

First, we show that there exist vertices $u_i, u_j \in V(C)$ with $3 \leqslant i \leqslant j-2$, $j \leqslant k-1$, such that the subgraphs $G(u_1, u_i, x)$ and $G(u_1, u_j, x)$ are triangles in G. Then, we prove that $i = 3$ and $j = k-1$.

By Observation 1 and Claim 3, there exists a triangle $G(u_1, x, y)$ such that $y \in V(C)$. Let $y = u_i$. By Claim 1, $x \not\sim \{u_2, u_k\}$. Hence, $3 \leqslant i \leqslant k-1$. Since $\deg x \geqslant 3$ and $G(N(x))$ is connected, there exists a vertex $z \in N(x)$ adjacent either to u_1 or to u_i. By symmetry, we can assume that $z \sim u_1$. By Claim 3, we have that $z \in V(C)$, i.e., $z = u_j$ for some $3 \leqslant j \neq i \leqslant k-1$. By symmetry, assume without loss of generality that $i < j$. Note that $j \neq i+1$ by Claim 1 and hence, $3 \leqslant i \leqslant j-2$, $j \leqslant k-1$. Thus we have two triangles $G(u_1, u_i, x)$ and $G(u_1, u_j, x)$.

Now we prove that $i = 3$ and $j = k-1$. Assuming the contrary, by symmetry we have one of the following two cases: $i \geqslant 4$, $j = k-1$ and $i \geqslant 4$, $j \leqslant k-2$. In both cases we will arrive at a contradiction.

Case 1. Let $i \geqslant 4$ and $j = k-1$.

We can easily see that $u_2 \not\sim u_k$ since otherwise C can be extended to the cycle $xu_1u_ku_2Cu_{k-1}x$. Since $\deg u_1 \in \{5, 6\}$, we have either $G(N(u_1)) \cong P_5$ or $G(N(u_1)) \cong C_6$ and in any case $u_2 \sim u_i$. Similarly to the proof in Case 2 of Claim 3, we can conclude that $u_{i+1} \not\sim \{u_2, u_{i-1}\}$. Similarly to Claim 4, we have $u_i \sim \{u_1, u_2, u_{i-1}, u_{i+1}, x\}$ and we arrive at the same contradiction as in Claim 4.

Case 2. Let $i \geqslant 4$ and $j \leqslant k-2$.

We first show that $u_2 \not\sim u_i$. Indeed, otherwise $u_{i+1} \not\sim \{u_2, u_{i-1}\}$ by a similar argument as in Case 1. By Claim 1, $x \not\sim u_{i+1}$. Thus we have $u_i \sim \{u_1, u_2, u_{i-1}, u_{i+1}, x\}$ and $u_{i+1} \not\sim \{u_1, u_2, u_{i-1}, x\}$. Then $G(N(u_i))$ is not isomorphic to P_5 or C_6, which is a contradiction to $\deg u_i \geqslant 5$.

Now we show that $u_i \not\sim u_k$. Suppose, to the contrary, that $u_i \sim u_k$. Then $u_{i-1} \not\sim \{u_{i+1}, u_k\}$ since, if $u_{i-1} \sim u_{i+1}$, we can extend C to the cycle $xu_1Cu_{i-1}u_{i+1}Cu_ku_ix$ and, if $u_{i-1} \sim u_k$, we can extend C to the cycle $xu_1Cu_{i-1}u_k\overline{C}u_ix$. By Claim 1, $x \not\sim u_{i-1}$. Thus we have $u_i \sim \{u_1, u_{i-1}, u_{i+1}, u_k, x\}$ and $u_{i-1} \not\sim \{u_1, u_{i+1}, u_k, x\}$. Then $G(N(u_i))$ is not isomorphic to P_5 or C_6, which is a contradiction to $\deg u_i \geqslant 5$.

By symmetry, similarly to $u_i \not\sim \{u_2, u_k\}$, we can obtain $u_j \not\sim \{u_2, u_k\}$. By Claim 1, $x \not\sim \{u_2, u_k\}$. Hence, in $G(N(u_1))$ we have $\{u_2, u_k\} \not\sim \{u_i, u_j, x\}$, which is a contradiction to both $G(N(u_1)) \cong P_5$ and $G(N(u_1)) \cong C_6$. □

Claim 6. *Relation* $\deg u_1 \neq 6$ *holds.*

Proof. Suppose, to the contrary, that $\deg u_1 = 6$. From the proof of Claim 5, we know that $\deg x \geqslant 3$. Thus the subgraphs $G(u_1, u_3, x)$ and $G(u_1, u_{k-1}, x)$ are triangles in G. Hence, $u_1 \sim \{u_2, u_3, u_{k-1}, u_k, x\}$.

Let y be a neighbor of u_1 different from u_2, u_3, u_{k-1}, u_k and x. By Claim 3, we have $y \in V(C)$, i.e., $y = u_i$, $4 \leqslant i \leqslant k-2$. Since $\deg u_1 = 6$ implies $G(N(u_1)) \cong C_6$, we have $u_i \sim \{u_2, u_k\}$. Note that $i \neq 4$ since otherwise $\deg u_3 > 2$ in $G(N(u_1))$, in contradiction to $G(N(u_1)) \cong C_6$. By symmetry, we have $i \neq k-2$. Thus, $5 \leqslant i \leqslant k-3$ and $\deg u_i \geqslant 5$ in G.

Now we show that $u_{i-1} \not\sim \{u_2, u_{i+1}, u_k\}$. Indeed, if $u_{i-1} \sim u_2$, then the cycle C can be extended to $xu_3Cu_{i-1}u_2u_iCu_1x$. If $u_{i-1} \sim u_{i+1}$, then the cycle $xu_3Cu_{i-1}u_{i+1}Cu_ku_iu_2u_1x$ is an extension of C, and if $u_{i-1} \sim u_k$, then the cycle $xu_1Cu_{i-1}u_ku_iCu_{k-1}x$ is an extension of C. Hence, in $G(N(u_i))$ we have $u_{i-1} \not\sim \{u_1, u_2, u_{i+1}, u_k\}$, and $G(N(u_i))$ is not isomorphic to P_5 or C_6. This is a contradiction to $\deg u_i \geqslant 5$. □

Claim 7. *Relations* $\deg u_1 = 5$ *and* $2 \leqslant \deg x \leqslant 6$ *hold.*

Proof. Since $3 \leqslant \deg u_1 \leqslant 6$, we have $\deg u_1 = 5$ from Claims 2, 4 and 6. According to the condition of the theorem, the subgraph $G(N(x))$ is isomorphic to one of the graphs P_2, P_3, P_4, P_5, or C_6, and therefore $2 \leqslant \deg x \leqslant 6$ follows immediately. □

Claim 8. *If* $\deg x = 2$*, then subgraph* $G(u_1, u_3, x)$ *is a triangle in* G*.*

Proof. By Observation 1 and Claim 3, there exists a triangle $G(u_1, x, y)$ such that $y \in V(C)$. Let $y = u_j$. By Claim 1, $x \not\sim \{u_2, u_k\}$. Hence, $3 \leqslant j \leqslant k-1$. Since $\deg x = 2$, the edge u_1x is contained only in one triangle $G(u_1, u_j, x)$. On the other hand, the edge u_1u_j is contained in at least two triangles $G(u_1, u_j, x)$ and $G(u_1, u_j, y)$ (with $y \neq x$), by Observation 2 because of $\deg u_1 = 5$. By Claim 3, $y \in V(C)$, say $y = u_i$. We can assume, by symmetry, that $2 \leqslant i \leqslant j-1$. Then, we prove that the only possibility is $i = 2$ and $j = 3$.

Assuming the contrary, we have one of the following three cases and in each of them we arrive at a contradiction.

Case 1. $i = 2$, $j \geqslant 4$.

Note that $u_j \sim \{u_1, u_2, u_{j-1}, u_{j+1}, x\}$ and, since $\deg x = 2$, the vertex x is an end vertex of $G(N(u_j))$. Therefore $G(N(u_j)) \cong P_5$ and the edge $u_{j-1}u_{j+1}$ is in P_5. Then $xu_ju_2Cu_{j-1}u_{j+1}Cu_1x$ is an extension of C.

Case 2. $i = j-1$, $j \geqslant 4$.

Note that $u_1 \sim \{u_2, u_{j-1}, u_j, u_k, x\}$ and therefore $G(N(u_1)) \cong P_5$. Since $\deg x = 2$, the vertex x is an end vertex of P_5 and therefore the edge u_2u_k is in P_5. Then $xu_jCu_ku_2Cu_{j-1}u_1x$ is an extension of C.

Case 3. $2 < i < j-1$, $j \geqslant 5$.

Note that $j+1 \neq k$ since otherwise $\deg u_1 > 2$ in $G(N(u_j))$ and we arrive at a contradiction to the condition of the theorem. On the other hand, $u_1 \sim \{u_2, u_i, u_j, u_k, x\}$ and $u_j \sim \{u_1, u_i, u_{j-1}, u_{j+1}, x\}$. Since $\deg x = 2$, the vertex x is an end vertex of both $G(N(u_1))$ and $G(N(u_j))$. Therefore $G(N(u_1)) \cong P_5$ and $G(N(u_j)) \cong P_5$. Hence, the graphs $G(N(u_1))$ and $G(N(u_j))$ contain the edges u_2u_k and $u_{j-1}u_{j+1}$, respectively.

There are four possibilities to get $G(N(u_1)) \cong P_5$ and $G(N(u_j)) \cong P_5$: (a) $u_i \sim u_2$, $u_i \sim u_{j-1}$; (b) $u_i \sim u_2$, $u_i \sim u_{j+1}$; (c) $u_i \sim u_k$, $u_i \sim u_{j-1}$; (d) $u_i \sim u_k$, $u_i \sim u_{j+1}$. We consider only the first one. The proofs of the other subcases are similar.

Note that $u_i \sim \{u_1, u_2, u_{i-1}, u_{i+1}, u_{j-1}, u_j\}$. Since $\deg u_i = 6$, we have $G(N(u_i)) \cong C_6$ and therefore it is clear that $u_{i-1} \sim u_{i+1}$. Then $xu_jCu_ku_2Cu_{i-1}u_{i+1}Cu_{j-1}u_iu_1x$ is an extension of C.

Note that in the proofs of subcases (b), (c) and (d), we also have $u_{i-1} \sim u_{i+1}$ and arrive at the following extensions of C: $xu_1u_iu_{j+1}Cu_ku_2Cu_{i-1}u_{i+1}Cu_jx$ in subcase (b) and $xu_1Cu_{i-1}u_{i+1}Cu_{j-1}u_{j+1}Cu_ku_iu_jx$ in subcases (c) and (d). □

In the proofs of the following claims, the vertex u of G will be called *completed* if, at the current step of the proof, we have constructed $G(N(u))$ and therefore we can definitely say to which of the graphs P_2, P_3, P_4, P_5, or C_6 the subgraph $G(N(u))$ is isomorphic.

Claim 9. *If* $\deg x = 2$*, then* $\deg u_3 = 5$*, the length* k *of the cycle* C *is at least* 8 *and the subgraph of* G *induced on the set of vertices* $\{u_1, u_2, u_3, u_4, u_5, u_{k-1}, u_k, x\}$ *has the following set of edges:* $\{u_1u_2, u_1u_3, u_1u_{k-1}, u_1u_k, u_1x, u_2u_3, u_2u_5, u_2u_{k-1}, u_3u_4, u_3u_5, u_3x, u_4u_5, u_{k-1}u_k\}$ (*see* Fig. 7).

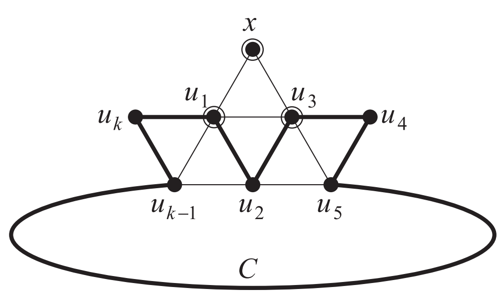

Fig. 7. Cycle C and graph $G(u_1, u_2, u_3, u_4, u_5, u_{k-1}, u_k, x)$

Proof. By Claim 8, the subgraph $G(u_1, u_3, x)$ is a triangle in G. By Claim 7, $\deg u_1 = 5$ and therefore there exists a neighbor y of u_1 different from u_2, u_3, u_k and x. By Claim 3, we have $y \in V(C)$, i.e., $y = u_i$, $4 \leqslant i \leqslant k-1$. Note that $i \neq 4$ since otherwise $\deg u_3 > 2$ in $G(N(u_1))$ and we arrive at a contradiction to the condition of the theorem. Let us show that $i = k-1$. Assuming the contrary, we have $5 \leqslant i \leqslant k-2$. Since $\deg u_1 = 5$, we have $G(N(u_1)) \cong P_5$ and since $\deg x = 2$, the vertex x is an end vertex of $G(N(u_1))$. The edges xu_3, u_2u_3 and u_iu_k are in $G(N(u_1))$ and we have two possibilities to get $G(N(u_1)) \cong P_5$: either $u_2 \sim u_k$ or $u_2 \sim u_i$. If $u_2 \sim u_k$, then $xu_3Cu_ku_2u_1x$ is an extension of C. Let $u_2 \sim u_i$. We have $u_i \sim \{u_1, u_2, u_{i-1}, u_{i+1}, u_k\}$. Now we can easily see that $u_{i-1} \not\sim \{u_2, u_{i+1}, u_k\}$ since otherwise C can be extended to the cycle $xu_3Cu_{i-1}u_2u_iCu_1x$ if $u_{i-1} \sim u_2$, to the cycle $xu_3Cu_{i-1}u_{i+1}Cu_ku_iu_2u_1x$ if $u_{i-1} \sim u_{i+1}$, and to the cycle $xu_3Cu_{i-1}u_k\overline{C}u_iu_2u_1x$ if $u_{i-1} \sim u_k$. Hence, in $G(N(u_i))$ we have $u_{i-1} \not\sim \{u_2, u_{i+1}, u_k\}$, which is a contradiction to both $G(N(u_i)) \cong P_5$ and $G(N(u_i)) \cong C_6$.

Thus, $i = k-1$. Since $G(N(u_1)) \cong P_5$ and $u_2 \not\sim u_k$, we have $u_2 \sim u_{k-1}$.

Note that $k \geqslant 6$, since otherwise $k = 5$ and $\deg u_1 > 2$ in $G(N(u_3))$. We arrive at a contradiction to the condition of the theorem. Hence, there exists a vertex u_4 different from u_{k-1}. Let us show that $\deg u_3 = 5$. Indeed, if $\deg u_3 = 4$, then we have $G(N(u_3)) \cong P_4$ and therefore $u_2 \sim u_4$. In this case, the cycle $xu_3u_2u_4Cu_1x$ is an extension of C. Moreover, $\deg u_3 \neq 6$ since the vertex x is an end vertex of $G(N(u_3))$ (because of $\deg x = 2$) and $G(N(u_3))$ cannot be isomorphic to C_6. Thus, $\deg u_3 = 5$ and $G(N(u_3)) \cong P_5$.

Let z be a neighbor of u_3 different from u_1, u_2, u_4 and x. If $z \in S$, then by Claim 1, we have $z \not\sim \{u_2, u_4\}$. In this case, $G(N(u_3))$ cannot be isomorphic to P_5. Hence, $z \in V(C)$, i.e., $z = u_j$. Note that $j \neq k-1$ and $j \neq k$ since otherwise $G(N(u_3))$ is not isomorphic to P_5. Thus, $5 \leqslant j \leqslant k-2$.

Let us show that $j = 5$. Suppose that $j \neq 5$. Then $6 \leqslant j \leqslant k-2$. Since $u_2 \not\sim u_4$ (as we already saw) and $G(N(u_3)) \cong P_5$, we have $u_j \sim \{u_2, u_4\}$. Now we can see that C can be extended to the cycle $xu_3Cu_ju_2u_{j+1}Cu_1x$ if $u_{j+1} \sim u_2$, to the cycle $xu_3u_2u_j\overline{C}u_4u_{j+1}Cu_1x$ if $u_{j+1} \sim u_4$ and to the cycle $xu_3u_2u_ju_4Cu_{j-1}u_{j+1}Cu_1x$ if $u_{j-1} \sim u_{j+1}$. Hence, in $G(N(u_j))$ we have $u_{j+1} \not\sim \{u_2, u_4, u_{j-1}\}$, which is a contradiction to both $G(N(u_j)) \cong P_5$ and $G(N(u_j)) \cong C_6$. Thus, $j = 5$ and $u_3 \sim u_5$.

To get $G(N(u_3)) \cong P_5$, we have the only possibility: $u_2 \sim u_5$. Thus we see that the set of edges of $G(u_1, u_2, u_3, u_4, u_5, u_{k-1}, u_k, x)$ includes all edges mentioned in Claim 9 (see Fig. 7). In Fig. 7, cycle C is given by the thick line, and the completed vertices are encircled (all edges incident to these vertices in G are shown in Fig. 7). To finish the proof, we have to show that the subgraph of G on the set of vertices $\{u_1, u_2, u_3, u_4, u_5, u_{k-1}, u_k, x\}$ is an induced subgraph, i.e., G has no edges u_4u_k, u_4u_{k-1}, u_5u_{k-1} and u_5u_k (we already saw that G has no edges u_2u_4 and u_2u_k). If $u_4 \sim u_k$, then $xu_1u_ku_4Cu_{k-1}u_2u_3x$ is an extension of C. Suppose that $u_5 \sim u_k$. If $k = 7$, we have the extension $xu_3u_4u_5u_2u_{k-1}u_ku_1x$ of C. Hence, $k > 7$ and there exists a vertex u_6 in C different from u_{k-1}. If $u_k \sim u_6$, then $xu_3u_4u_5u_ku_6Cu_{k-1}u_2u_1x$ is an extension of C. Therefore, in $G(N(u_5))$ we have $u_k \not\sim \{u_2, u_4, u_6\}$, which is a contradiction to both $G(N(u_5)) \cong P_5$ and $G(N(u_5)) \cong C_6$. Hence, $u_5 \not\sim u_k$ and by symmetry we have $u_4 \not\sim u_{k-1}$. If $u_5 \sim u_{k-1}$, then we have a cycle on four vertices in $G(N(u_2))$ and arrive at a contradiction to the condition of the theorem.

Finally, since the subgraph of G on the set of vertices $\{u_1, u_2, u_3, u_4, u_5, u_{k-1}, u_k, x\}$ is an induced subgraph, we have $k \geqslant 8$. □

Claim 10. *If* $\deg x = 2$, *then* $\deg u_2 \notin \{4, 5\}$.

Proof. Assume to the contrary that $\deg u_2 \in \{4, 5\}$. Starting from the construction of Claim 9 (see Fig. 7), we first show that $k \geqslant 9$ and specify the structure of graph G.

From the proof of Claim 9 we know that the vertices x, u_1 and u_3 are completed. If $\deg u_2 = 4$, the vertex u_2 is also completed since $G(N(u_2)) \cong P_4$. By Claim 9, $k \geqslant 8$ and there exists a vertex u_6 in C. Note that $u_4 \not\sim u_6$ since otherwise $xu_3u_2u_5u_4u_6Cu_1x$ is an extension of C. The vertex u_2 is an end vertex of $G(N(u_5))$ and therefore $G(N(u_5)) \cong P_5$. Let y be a neighbor of u_5 different from u_2, u_3, u_4 and u_6. If $y \in S$, then by Claim 1, we have $y \not\sim \{u_4, u_6\}$. In this case, $G(N(u_5))$ cannot be isomorphic to P_5. Hence, $y \in V(C)$, i.e., $y = u_i$, $7 \leqslant i \leqslant k-2$. Let us show that $i = 7$. Assuming the contrary, we have $8 \leqslant i \leqslant k-2$. To get $G(N(u_5)) \cong P_5$, we have the only possibility: $u_i \sim \{u_4, u_6\}$. Now we can easily see that $u_{i+1} \not\sim \{u_4, u_6, u_{i-1}\}$ since otherwise C can be extended to the cycle $xu_3u_2u_5Cu_iu_4u_{i+1}Cu_1x$ if $u_{i+1} \sim u_4$, to the cycle $xu_3u_2u_5u_4u_i\overline{C}u_6u_{i+1}Cu_1x$ if $u_{i+1} \sim u_6$ and to the cycle $xu_3u_2u_5u_4u_iu_6Cu_{i-1}u_{i+1}Cu_1x$ if $u_{i+1} \sim u_{i-1}$. Hence, in $G(N(u_i))$ we have $u_{i+1} \not\sim \{u_4, u_6, u_{i-1}\}$, which is a contradiction to both $G(N(u_i)) \cong P_5$ and $G(N(u_i)) \cong C_6$. Thus, $i = 7$, $u_4 \sim u_7$ and the vertex u_5 is completed (see Fig. 8). Note that $\deg u_4 \neq 6$ since the completed vertex u_3 is an end vertex of the subgraph $G(N(u_4))$, which is therefore not isomorphic to C_6. From $i = 7$, we have $k \geqslant 9$.

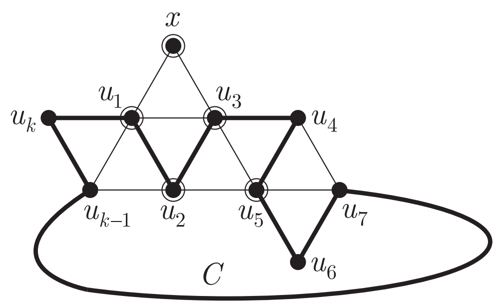

Fig. 8. A part of graph G for the case of $\deg u_2 = 4$

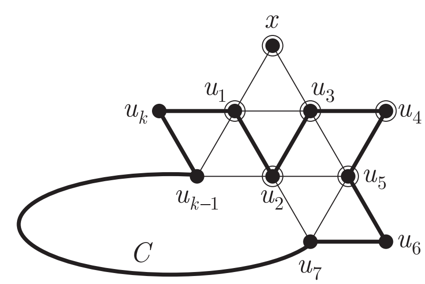

Fig. 9. A part of graph G for the case of $\deg u_2 = 5$

If $\deg u_2 = 5$, the vertex u_2 is not completed yet. Let y be a neighbor of u_2 different from u_1, u_3, u_5 and u_{k-1}. It is easy to see that $y \in V(C)$, i.e., $y = u_j$, $6 \leqslant j \leqslant k-2$. Moreover, if $j = 6$, the cycle C can be extended to $xu_3u_4u_5u_2u_6Cu_1x$. The case when $j = k-2$ is symmetric. Thus, $7 \leqslant j \leqslant k-3$. Since $G(N(u_2)) \cong P_5$, we can assume without loss of generality that $u_5 \sim u_j$. Now the vertex u_2 is completed. Let us show that $j = 7$. Assuming the contrary, we have $8 \leqslant j \leqslant k-3$. We have $u_4 \not\sim u_6$ (as in the previous case) and either $G(N(u_5)) \cong P_5$ or $G(N(u_5)) \cong C_6$. Hence, $u_6 \sim u_j$. The completed vertex u_2 is an end vertex of $G(N(u_j))$ and therefore $G(N(u_j)) \cong P_5$. Now we show that $u_{j+1} \not\sim \{u_6, u_{j-1}\}$. Indeed, if $u_6 \sim u_{j+1}$, then the cycle $xu_3u_4u_5u_2u_j\overline{C}u_6u_{j+1}Cu_1x$ is an extension of C. If $u_{j-1} \sim u_{j+1}$, then the cycle $xu_3u_4u_5u_2u_ju_6Cu_{j-1}u_{j+1}Cu_1x$ is an extension of C. Hence, $G(N(u_j))$ cannot be isomorphic to P_5 and we arrive at a contradiction. Thus, $j = 7$ and $u_7 \sim \{u_2, u_5\}$. Since $j = 7$, we have $k \geqslant 9$. Now we show that vertices u_4 and u_5 are completed (see Fig. 9). Indeed, $G(N(u_5))$ cannot be isomorphic to C_6 since otherwise there exists a vertex z such that $z \sim \{u_4, u_6\}$. By Claim 1, $z \in V(C)$, i.e., $z = u_t$, $8 \leqslant t \leqslant k-2$. Similarly to previous arguments, by considering $G(N(u_t))$, one can show that $9 \leqslant t \leqslant k-3$ and $u_{t+1} \not\sim \{u_4, u_6, u_{t-1}\}$. Hence, $G(N(u_t))$ is not isomorphic to P_5 or C_6, which is a contradiction to the condition of the theorem. Therefore, $G(N(u_5)) \cong P_5$ and the vertex u_5 is completed. As a consequence, u_4 is also completed and $G(N(u_4)) \cong P_2$. Indeed, its neighbors u_3 and u_5 are completed and, moreover, $u_4 \not\sim u_7$ (otherwise $G(N(u_5))$ is not isomorphic to P_5) and $u_4 \not\sim u_6$.

Let $l = \lceil k/2 \rceil - 4$, where k is the length of C. Since $k \geqslant 9$, we have $l \geqslant 1$. To complete the proof, in l steps we arrive at a contradiction with the non-extendability of the cycle C.

Let p denote the number of a step, $1 \leqslant p \leqslant l$. In the first step (when $p = 1$), the following is valid for both cases $\deg u_2 = 4$ and $\deg u_2 = 5$ (see Fig. 8 and Fig. 9):

(a) there is an (x, u_7)-path in G with the vertex set $\{x, u_7\} \cup \{u_2, u_3, u_4, u_5\}$ (if $\deg u_2 = 4$, the path is $xu_3u_2u_5u_4u_7$, and if $\deg u_2 = 5$, the path is $xu_3u_4u_5u_2u_7$);
(b) there is a unique index r such that $r \in \{2p-2, 2p, 2p+2\}$ and $u_r \sim \{u_{2p+3}, u_{2p+5}\}$ (in particular, we have $r = 4$, $u_4 \sim \{u_5, u_7\}$ if $\deg u_2 = 4$, and $r = 2$, $u_2 \sim \{u_5, u_7\}$ if $\deg u_2 = 5$);
(c) all vertices $u_2, u_3, u_4, \ldots, u_{2p+3}$ are completed with the possible exception of u_r (in particular, from the sequence of completed vertices we have an exception of u_4 if $\deg u_2 = 4$, and we do not have any exceptions if $\deg u_2 = 5$);
(d) $u_5 \sim u_7$, $\deg u_5 = 5$ and $\deg u_i \neq 6$, $2 \leqslant i \leqslant 5$.

In what follows, we use an induction on the number of steps p and show that the following construction will be valid in step $p+1$ if it is valid in step p for $1 \leqslant p < l$:

(a) there is an (x, u_{2p+5})-path Q_p in G with the vertex set $\{x, u_{2p+5}\} \cup \{u_2, u_3, \ldots, u_{2p+3}\}$;
(b) there is a unique index r such that $r \in \{2p-2, 2p, 2p+2\}$ and $u_r \sim \{u_{2p+3}, u_{2p+5}\}$;
(c) all vertices $u_2, u_3, u_4, \ldots, u_{2p+3}$ are completed with the possible exception of u_r;
(d) $u_{2p+3} \sim u_{2p+5}$, $\deg u_{2p+3} = 5$ and $\deg u_i \neq 6$, $2 \leqslant i \leqslant 2p+3$.

Obviously, this construction is valid in the first step. Suppose that it is valid in step p and consider the possible three cases for the index r, $r \in \{2p-2, 2p, 2p+2\}$.

Case 1. $r = 2p-2$.

Note that $u_{2p-2} \sim \{u_{2p+3}, u_{2p+5}\}$. Moreover, $u_{2p-2} \sim u_{2p+1}$. Indeed, in the previous step there is a unique index $s \in \{2p-4, 2p-2, 2p\}$ such that $u_s \sim \{u_{2p+1}, u_{2p+3}\}$. Since $u_{2p-2} \sim u_{2p+3}$ and the vertex u_{2p+3} is completed, we have $s = 2p-2$ and therefore $u_{2p-2} \sim u_{2p+1}$. Now the vertex u_{2p-2} is completed since $\deg u_{2p-2} = 5$ and $G(N(u_{2p-2})) \cong P_5$.

First, we show that $\deg u_{2p+5} = 5$. If $u_{2p+4} \sim u_{2p+6}$, then $Q_p u_{2p+4} u_{2p+6} C u_1 x$ is an extension of C. Hence, $G(N(u_{2p+5}))$ is not isomorphic to P_4. The vertex u_{2p-2} is an end vertex of $G(N(u_{2p+5}))$ and therefore $G(N(u_{2p+5})) \cong P_5$ and $\deg u_{2p+5} = 5$. Let y be a neighbor of u_{2p+5} different from u_{2p-2}, u_{2p+3}, u_{2p+4} and u_{2p+6}. If $y \in S$, then by Claim 1, we have $y \not\sim \{u_{2p+4}, u_{2p+6}\}$. In this case, $G(N(u_{2p+5}))$ cannot be isomorphic to P_5. Hence, $y \in V(C)$, i.e., $y = u_t$, $2p+7 \leqslant t \leqslant k$. Let us show that $t = 2p+7$. Assuming the contrary, we have $2p+8 \leqslant t \leqslant k$. To get $G(N(u_{2p+5})) \cong P_5$, we have the only possibility: $u_t \sim \{u_{2p+4}, u_{2p+6}\}$. Now we can easily see that $u_{t+1} \not\sim \{u_{2p+4}, u_{2p+6}, u_{t-1}\}$ since otherwise C can be extended to the cycle $Q_p u_{2p+6} C u_t u_{2p+4} u_{t+1} C u_1 x$ if $u_{t+1} \sim u_{2p+4}$, to the cycle $Q_p u_{2p+4} u_t \overline{C} u_{2p+6} u_{t+1} C u_1 x$ if $u_{t+1} \sim u_{2p+6}$ and to the cycle $Q_p u_{2p+4} u_t u_{2p+6} C u_{t-1} u_{t+1} C u_1 x$ if $u_{t+1} \sim u_{t-1}$. Hence, in $G(N(u_t))$ we have $u_{t+1} \not\sim \{u_{2p+4}, u_{2p+6}, u_{t-1}\}$, which is a contradiction to both $G(N(u_t)) \cong P_5$ and $G(N(u_t)) \cong C_6$. Thus, $t = 2p+7$, $u_{2p+4} \sim u_{2p+7}$ and the vertex u_{2p+5} is completed.

Now we have the (x, u_{2p+7})-path $Q_p u_{2p+4} u_{2p+7}$ in G and the index $2p+4$ from the set $\{2p, 2p+2, 2p+4\}$ with $u_{2p+4} \sim \{u_{2p+5}, u_{2p+7}\}$. Note that $u_{2p+5} \not\sim \{u_{2p}, u_{2p+2}\}$ since the vertex u_{2p+5} is already completed and therefore $2p+4$ is the unique index with $u_{2p+4} \sim \{u_{2p+5}, u_{2p+7}\}$. Moreover, all vertices $u_2, u_3, u_4, \ldots, u_{2p+5}$ are completed with the exception of u_{2p+4}. Note that $\deg u_{2p+4} \neq 6$ since the completed vertex u_{2p+3} is an end vertex of the subgraph $G(N(u_{2p+4}))$ which is therefore not isomorphic to C_6. Hence, $\deg u_i \neq 6$, $2 \leqslant i \leqslant 2p+5$. We already saw that $u_{2p+5} \sim u_{2p+7}$ and $\deg u_{2p+5} = 5$. Thus all properties (a) – (d) are valid in step $p+1$.

Case 2. $r = 2p$.

In this case, we have $u_{2p} \sim \{u_{2p+3}, u_{2p+5}\}$. Since all vertices $u_2, u_3, u_4, \dots, u_{2p+3}$ are completed with the possible exception of u_{2p}, we have two possibilities: either the vertex u_{2p} is completed or, if it is not completed, the subgraph $G(N(u_{2p}))$ is isomorphic to P_5 because of $\deg u_{2p} \neq 6$.

In the first variant, $\deg u_{2p} = 4$ and $G(N(u_{2p})) \cong P_4$. Similarly to the previous case, one can show that $\deg u_{2p+5} = 5$, $u_{2p+4} \sim u_{2p+7}$, $u_{2p+5} \sim u_{2p+7}$ and the vertex u_{2p+5} is completed. Moreover all properties (a) – (d) are valid in step $p+1$ as in the previous case. The only difference is the following. To show that $2p+4$ is the unique index from the set $\{2p, 2p+2, 2p+4\}$ with $u_{2p+4} \sim \{u_{2p+5}, u_{2p+7}\}$, one can use that $u_{2p} \not\sim u_{2p+7}$ and $u_{2p+2} \not\sim u_{2p+5}$ (since the vertices u_{2p} and u_{2p+5} are completed).

In the second variant, $\deg u_{2p} = 5$. Let y be a neighbor of u_{2p} different from u_{2p-1}, u_{2p+1}, u_{2p+3} and u_{2p+5}. To get $G(N(u_{2p})) \cong P_5$, we have the only possibility $y \sim u_{2p+5}$. If $y \in S$, then by Claim 1, we have $y \not\sim \{u_{2p+4}, u_{2p+6}\}$. Moreover, as in the previous case, $u_{2p+4} \not\sim u_{2p+6}$. Hence, $G(N(u_{2p+5}))$ cannot be isomorphic to P_5 or C_6. Therefore, $y \in V(C)$, i.e., $y = u_t$, $2p+7 \leqslant t \leqslant k$. On the other hand, $u_t \sim u_{2p+6}$ for both cases $G(N(u_{2p+5})) \cong P_5$ or $G(N(u_{2p+5})) \cong C_6$. Let us show that $t = 2p+7$. Assuming the contrary, we have $2p+8 \leqslant t \leqslant k$. Now we see that $u_{t+1} \not\sim \{u_{2p+6}, u_{t-1}\}$ since otherwise C can be extended to the cycle $Q_{p-2}u_{2p+2}u_{2p+3}u_{2p+4}u_{2p+5}u_{2p}u_t\overline{C}u_{2p+6}u_{t+1}Cu_1x$ if $u_{t+1} \sim u_{2p+6}$ and to the cycle $Q_{p-2}u_{2p+2}u_{2p+3}u_{2p+4}u_{2p+5}u_{2p}u_tu_{2p+6}Cu_{t-1}u_{t+1}Cu_1x$ if $u_{t+1} \sim u_{t-1}$. Here Q_{p-2} is the (x, u_{2p+1})-path obtained in step $p-2$ according to property (a). Moreover, $u_{t+1} \not\sim u_{2p}$ since $\deg u_{2p} = 5$. Hence, in $G(N(u_t))$ we have $u_{t+1} \not\sim \{u_{2p}, u_{2p+6}, u_{t-1}\}$, which is a contradiction to both $G(N(u_t)) \cong P_5$ and $G(N(u_t)) \cong C_6$. Thus, $t = 2p+7$, $u_{2p+5} \sim u_{2p+7}$ and the vertex u_{2p} is completed.

Let us show that $\deg u_{2p+5} = 5$. If $\deg u_{2p+5} = 6$, there is a neighbor y of u_{2p+5} different from u_{2p}, u_{2p+3}, u_{2p+4}, u_{2p+6}, u_{2p+7} and such that $y \sim \{u_{2p+4}, u_{2p+6}\}$. By Claim 1, $y \in V(C)$, i.e., $y = u_s$, $2p+8 \leqslant s \leqslant k$. Note that $s \neq 2p+8$ since otherwise $\deg u_{2p+7} > 2$ in $G(N(u_{2p+5}))$, in contradiction to $G(N(u_{2p+5})) \cong C_6$. Thus, $2p+9 \leqslant s \leqslant k$. Using similar arguments as in Case 1 for the vertex u_{t+1}, we can show that $u_{s+1} \not\sim \{u_{2p+4}, u_{2p+6}, u_{s-1}\}$ which is a contradiction to both $G(N(u_s)) \cong P_5$ and $G(N(u_s)) \cong C_6$. Therefore, $\deg u_{2p+5} = 5$ and the vertex u_{2p+5} is completed.

Now we have the (x, u_{2p+7})-path $Q_{p-2}u_{2p+2}u_{2p+3}u_{2p+4}u_{2p+5}u_{2p}u_{2p+7}$ in G and the index $2p$ from the set $\{2p, 2p+2, 2p+4\}$ with $u_{2p} \sim \{u_{2p+5}, u_{2p+7}\}$. Note that $u_{2p+4} \not\sim \{u_{2p+6}, u_{2p+7}\}$ since the vertex u_{2p+5} is already completed and therefore $2p$ is the unique index with $u_{2p} \sim \{u_{2p+5}, u_{2p+7}\}$ and the vertex u_{2p+4} is completed. Now all vertices $u_2, u_3, u_4, \dots, u_{2p+5}$ are completed. Note that $\deg u_{2p+4} = 2$ and $\deg u_i \neq 6$, $2 \leqslant i \leqslant 2p+5$. We already saw that $u_{2p+5} \sim u_{2p+7}$ and $\deg u_{2p+5} = 5$. Thus all properties (a) – (d) are valid in step $p+1$.

Case 3. $r = 2p+2$.

In this case, we have $u_{2p+2} \sim \{u_{2p+3}, u_{2p+5}\}$. Since all vertices $u_2, u_3, u_4, \dots, u_{2p+3}$ are completed with the possible exception of u_{2p+2}, we have two variants: either the vertex u_{2p+2} is completed or not.

In the first variant, $\deg u_{2p+2} = 3$ and $G(N(u_{2p+2})) \cong P_3$. Similarly to Case 1, one can show that $\deg u_{2p+5} = 5$, $u_{2p+4} \sim u_{2p+7}$, $u_{2p+5} \sim u_{2p+7}$ and the vertex u_{2p+5} is completed. Moreover all properties (a) – (d) are valid in step $p+1$ as in Case 1. The only difference is the following. To show that $2p+4$ is the unique index from the set $\{2p, 2p+2, 2p+4\}$ with $u_{2p+4} \sim \{u_{2p+5}, u_{2p+7}\}$, one can use that $u_{2p+2} \not\sim u_{2p+7}$ and $u_{2p} \not\sim u_{2p+5}$ (since the vertices u_{2p+2} and u_{2p+5} are completed).

In the second variant, vertex u_{2p+2} is not completed and there exists a neighbor y of u_{2p+2} different from u_{2p+1}, u_{2p+3}, u_{2p+5} and such that $y \sim u_{2p+5}$. If $y \in S$, then by Claim 1, we have $y \not\sim u_{2p+6}$. Moreover, $u_{2p+4} \not\sim u_{2p+6}$ since otherwise $Q_p u_{2p+4} u_{2p+6} C u_1 x$ is an extension of C. Hence, $G(N(u_{2p+5}))$ is not isomorphic to P_5 or C_6, which is a contradiction to the condition of the theorem. Therefore, $y \in V(C)$, i.e., $y = u_t$, $2p+6 \leqslant t \leqslant k$. Clearly, $t \neq 2p+6$ since otherwise $Q_{p-1} u_{2p+4} u_{2p+5} u_{2p+2} u_{2p+6} C u_1 x$ is an extension of C. Here Q_{p-1} is the (x, u_{2p+3})-path obtained in step $p-1$ according to the property (a). Therefore, $2p+7 \leqslant t \leqslant k$. Let us show that $t = 2p+7$. Assuming the contrary, we have $2p+8 \leqslant t \leqslant k$. Since $u_{2p+4} \not\sim u_{2p+6}$ and $G(N(u_{2p+5}))$ is isomorphic to P_5 or C_6, we have $u_t \sim u_{2p+6}$. Now we see that $u_{t+1} \not\sim \{u_{2p+2}, u_{2p+6}, u_{t-1}\}$ since otherwise C can be extended to the cycle $Q_{p-1} C u_t u_{2p+2} u_{t+1} C u_1 x$ if $u_{t+1} \sim u_{2p+2}$, to the cycle $Q_{p-1} u_{2p+4} u_{2p+5} u_{2p+2} u_t \overline{C} u_{2p+6} u_{t+1} C u_1 x$ if $u_{t+1} \sim u_{2p+6}$, and to the cycle $Q_{p-1} u_{2p+4} u_{2p+5} u_{2p+2} u_t u_{2p+6} C u_{t-1} u_{t+1} C u_1 x$ if $u_{t+1} \sim u_{t-1}$. Hence, in $G(N(u_t))$ we have $u_{t+1} \not\sim \{u_{2p+2}, u_{2p+6}, u_{t-1}\}$, which is a contradiction to both $G(N(u_t)) \cong P_5$ and $G(N(u_t)) \cong C_6$. Thus, $t = 2p+7$ and $u_{2p+7} \sim \{u_{2p+2}, u_{2p+5}\}$.

Now we show that vertices u_{2p+4} and u_{2p+5} are completed. Indeed, $G(N(u_{2p+5}))$ cannot be isomorphic to C_6 since otherwise there exists a vertex z such that $z \sim \{u_{2p+4}, u_{2p+6}\}$. By Claim 1, $z \in V(C)$, i.e., $z = u_s$, $2p+8 \leqslant s \leqslant k$. By considering $G(N(u_s))$, one can show that $2p+9 \leqslant s \leqslant k$ and $u_{s+1} \not\sim \{u_{2p+4}, u_{2p+6}, u_{s-1}\}$ (similarly to the previous argument with vertex $y = u_t$). Hence, $G(N(u_s))$ is not isomorphic to P_5 or C_6, which is a contradiction to the condition of the theorem. Therefore, $G(N(u_{2p+5})) \cong P_5$ and the vertex u_{2p+5} is completed. As a consequence, u_{2p+4} is also completed and $G(N(u_{2p+4})) \cong P_2$. Indeed, its neighbors u_{2p+3} and u_{2p+5} are completed and, moreover, $u_{2p+4} \not\sim u_{2p+2}$, $u_{2p+4} \not\sim u_{2p+7}$ (otherwise $G(N(u_{2p+5}))$ is not isomorphic to P_5) and $u_{2p+4} \not\sim u_{2p+6}$.

Now we have the (x, u_{2p+7})-path $Q_{p-1} u_{2p+4} u_{2p+5} u_{2p+2} u_{2p+7}$ in G and the index $2p+2$ from the set $\{2p, 2p+2, 2p+4\}$ with $u_{2p+2} \sim \{u_{2p+5}, u_{2p+7}\}$. All vertices $u_2, u_3, u_4, \ldots, u_{2p+5}$ are completed with the possible exception of u_{2p+2} and therefore $\deg u_{2p+4} = 2$, $\deg u_{2p+5} = 5$ and $2p+2$ is the unique index with $u_{2p+2} \sim \{u_{2p+5}, u_{2p+7}\}$. Hence, $\deg u_i \neq 6$, $2 \leqslant i \leqslant 2p+5$. We already saw that $u_{2p+5} \sim u_{2p+7}$. Thus all properties (a) – (d) are valid in step $p+1$.

Consequently, in all three cases we have shown that the properties (a) – (d) are valid in step $p+1$ if they are valid in step p for $1 \leqslant p < l$, where $l = \lceil k/2 \rceil - 4$ and k is the length of C. Let us show that the properties (a) – (d) in step l result in a contradiction with the non-extendability of the cycle C.

Suppose first that k is even, i.e., $2l = k-8$ and $u_{2l+6} = u_{k-2}$. We have $u_{2l+6} \not\sim \{u_r, u_{2l+4}\}$ since otherwise C can be extended to the cycle $QCu_{2l+5} u_r u_{2l+6} C u_1 x$ if $u_r \sim u_{2p+6}$ and to the cycle $Q_l u_{2l+4} u_{2l+6} C u_1 x$ if $u_{2l+4} \sim u_{2p+6}$. Here Q is the (x, u_{r+1})-path obtained according to the property (a) in step $l-3$ if $r = 2l-2$, in step $l-2$ if $r = 2l$, or in step $l-1$ if $r = 2l+2$. Since $u_{2l+6} \not\sim \{u_r, u_{2l+4}\}$, we have that $G(N(u_{2l+5}))$ cannot be isomorphic to P_4. Hence, there exists a neighbor y of u_{2l+5} different from u_r, u_{2l+3}, u_{2l+4} and u_{2l+6}. If $y \in S$, then by Claim 1, we have $y \not\sim \{u_{2l+4}, u_{2l+6}\}$, and $G(N(u_{2l+5}))$ is not isomorphic to P_5 or C_6, which is a contradiction to the condition of the theorem.

Therefore, $y \in V(C)$. Since all vertices $u_2, u_3, u_4, \dots, u_{2l+3}$ are completed with the possible exception of u_r, we have $y = u_{k-1}$ or $y = u_k$.

To get $G(N(u_{2l+5})) \cong P_5$ or $G(N(u_{2l+5})) \cong C_6$ in case of $y = u_{k-1}$, we have either $u_r \sim u_{k-1}$, or $u_{2l+4} \sim u_{k-1}$. Hence, $\deg u_{k-1} = 6$ and we arrive to contradiction with $G(N(u_{k-1})) \cong C_6$ since the end vertex u_2 of $G(N(u_{k-1}))$ is completed.

To get $G(N(u_{2l+5})) \cong P_5$ or $G(N(u_{2l+5})) \cong C_6$ in case of $y = u_k$, we have $u_{2l+6} \sim u_k$, and either $u_r \sim u_k$ or $u_{2l+4} \sim u_k$. Let $u_r \sim u_k$. Then cycle C can be extended to the cycle $QCu_{2l+5}u_ru_ku_{2l+6}u_{k-1}u_1x$. As before, Q is the (x, u_{r+1})-path obtained according to the property (a) in step $l-3$ if $r = 2l-2$, in step $l-2$ if $r = 2l$, or in step $l-1$ if $r = 2l+2$. Let $u_{2l+4} \sim u_k$. Then C can be extended to the cycle $xu_3Cu_{2l+4}u_ku_{2l+5}u_{2l+6}u_{k-1}u_2u_1x$.

Suppose now that k is odd, i.e., $2l = k-7$ and hence $u_{2l+6} = u_{k-1}$. In this case, we arrive at a contradiction with the non-extendability of cycle C in the same way as for even k.

Thus the assumption $\deg u_2 \in \{4, 5\}$ leads to a contradiction and the proof of the claim is completed. □

Claim 11. *If* $\deg x = 2$ *and* $\deg u_2 = 6$*, then* G *is isomorphic to graph* D*.*

Proof. Since $\deg u_2 = 6$, we have $G(N(u_2)) \cong C_6$ and there exists a vertex $y \in N(u_2)$ such that $y \sim u_5$ and y differs from u_3. Similarly to the proof of Claim 10 in case of $\deg u_2 = 5$, it is easy to show that $y = u_7$ and $k \geqslant 9$. Note that $k \geqslant 10$ since otherwise the cycle $xu_3Cu_7u_2u_{k-1}u_ku_1x$ is an extension of C. Hence $u_8 \neq u_{k-1}$. Since $G(N(u_2)) \cong C_6$, there exists a vertex $z \in N(u_2)$ such that $z \sim \{u_7, u_{k-1}\}$. If $z \in S$, then by Claim 1 we have $z \not\sim u_8$. Note that $u_6 \not\sim u_8$ since otherwise the cycle $xu_3u_4u_5u_2u_7u_6u_8Cu_{k-1}u_ku_1x$ is an extension of C. Therefore, in $G(N(u_7))$ we have $u_8 \not\sim \{u_6, z\}$, which is a contradiction to both $G(N(u_7)) \cong P_5$ and $G(N(u_7)) \cong C_6$. So we have $z \in V(C)$, i.e., $z = u_i$, $8 \leqslant i \leqslant k-2$. Moreover, if $i = 8$ or $i = k-2$, cycles $xu_3Cu_7u_2u_8Cu_1x$ or $xu_3Cu_{k-2}u_2u_{k-1}u_ku_1x$, respectively, are extensions of C. Thus $9 \leqslant i \leqslant k-3$.

Let us show that either $i = 9$ or $i = k-3$. Assuming the contrary, we have $10 \leqslant i \leqslant k-4$. Note that $u_{k-2} \not\sim u_k$ since otherwise the cycle $xu_3Cu_{k-2}u_ku_{k-1}u_2u_1x$ is an extension of C. Since graphs $G(N(u_7))$ and $G(N(u_{k-1}))$ are isomorphic either to P_5 or to C_6 and, moreover, $u_6 \not\sim u_8$, $u_{k-2} \not\sim u_k$, we have $u_i \sim \{u_8, u_{k-2}\}$. Thus $\deg u_i > 6$, which is a contradiction to the condition of the theorem.

Let $i = k-3$. Then $u_{k-3} \sim \{u_7, u_{k-1}\}$, $G(N(u_2)) \cong C_6$ and vertex u_2 is completed. The vertices u_1 and u_3 are completed according to the proof of Claim 9. Similarly to the proof of Claim 10 one can show that the vertices u_4 and u_5 are also completed (see Fig. 10, where cycle C is given by the thick line, and the completed vertices are encircled; all edges incident to these vertices in G are shown in the figure). Note that $k \geqslant 12$ since otherwise the cycle $xu_3Cu_7u_2u_{k-3}Cu_1x$ is an extension of C. Hence $u_8 \neq u_{k-3}$. Since $u_6 \not\sim u_8$ and graph $G(N(u_7))$ is isomorphic either to P_5 or to C_6, we have $u_8 \sim u_{k-3}$. Let us show that $k = 12$, i.e., $u_8 = u_{k-4}$. Assuming the contrary, we have $G(N(u_{k-3})) \cong C_6$ and therefore $u_{k-4} \sim \{u_8, u_{k-2}\}$. In this case, the cycle $xu_3Cu_{k-4}u_{k-2}u_{k-3}u_2u_{k-1}u_ku_1x$ is an extension of C.

Let $i = 9$. Similarly to the proof of the case $i = k-3$, we have that vertices u_1, u_2, u_3, u_4 and u_5 are completed (see Fig. 11). Note that $k \geqslant 12$ since otherwise the cycle

$xu_3Cu_9u_2u_{k-1}u_ku_1x$ is an extension of C. Hence $u_{10} \neq u_{k-2}$. If $u_8 \sim u_{10}$, the cycle C can be extended to the cycle $xu_3Cu_7u_2u_9u_8u_{10}Cu_ku_1x$. Since graph $G(N(u_9))$ is isomorphic either to P_5 or to C_6, we have $u_{10} \sim u_{k-1}$. Let us show that $k = 12$, i.e., $u_{10} = u_{k-2}$. Assuming the contrary, we have $G(N(u_{k-1})) \cong C_6$ and therefore $u_{k-2} \sim \{u_{10}, u_k\}$. In this case, the cycle $xu_3Cu_{k-2}u_ku_{k-1}u_2u_1x$ is an extension of C.

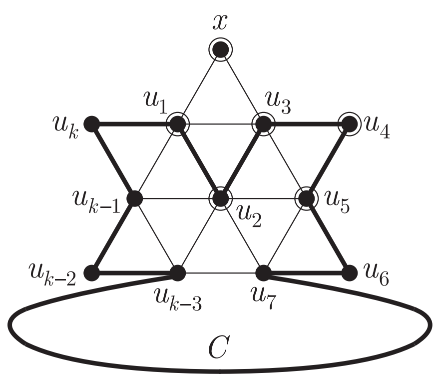

Fig. 10. A part of graph G for the case of $i = k - 3$

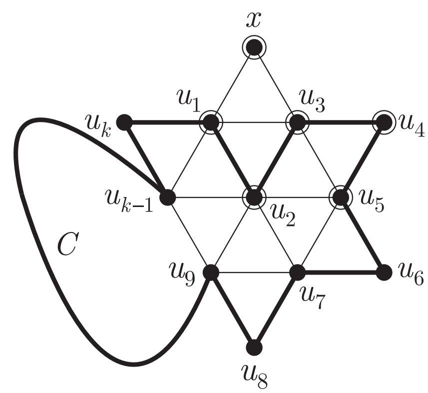

Fig. 11. A part of graph G for the case of $i = 9$

Thus in both cases $i = 9$ or $i = k - 3$, we have $C = u_1u_2u_3u_4u_5u_6u_7u_8u_9u_{10}u_{11}u_{12}$. The vertices u_7, u_9, u_{11} are completed. Assuming the contrary, say u_7 is not completed, we have $G(N(u_7)) \cong C_6$. Then there is a vertex y adjacent to u_7 such that $y \sim \{u_6, u_8\}$. By Claim 1, $y \in V(C)$. Then the only possibility is $y = u_{12}$ and cycle C can be extended to the cycle $xu_3u_4u_5u_6u_{12}u_7u_8u_9u_{10}u_{11}u_2u_1x$. The completeness of vertices u_9 and u_{11} is shown similarly.

Since $u_6 \not\sim u_8$, we have $G(N(u_6)) \cong P_2$ and the vertex u_6 is completed. Note that $u_{10} \not\sim \{u_8, u_{12}\}$ since otherwise $G(N(u_9)) \cong G(N(u_{11})) \cong C_5$, which is a contradiction to the condition of the theorem. Therefore graphs $G(N(u_8))$, $G(N(u_{10}))$ and $G(N(u_{12}))$ are isomorphic to P_2, and vertices u_8, u_{10}, u_{12} are completed.

Thus all vertices of graph G are completed and G is isomorphic to graph D. □

Claims 8 – 11 show that G is isomorphic to graph D if $\deg x = 2$. According to Claim 7, we have to consider the situation $3 \leqslant \deg x \leqslant 6$ for completing the proof of the theorem.

Claim 12. *Relation* $\deg x \neq 3$ *holds.*

Proof. Suppose, to the contrary, that $\deg x = 3$. From Claim 5, we know that subgraphs $G(u_1, u_3, x)$ and $G(u_1, u_{k-1}, x)$ are triangles in G. Then $\deg u_3 = 5$, the length k of the cycle C is at least 8 and the subgraph of G induced on the set of vertices $\{u_1, u_2, u_3, u_4, u_5, u_{k-1}, u_k, x\}$ (see Fig. 12) has the following set of edges:

$$\{u_1u_2, u_1u_3, u_1u_{k-1}, u_1u_k, u_1x, u_2u_3, u_2u_5, u_3u_4, u_3u_5, u_3x, u_4u_5, u_{k-1}u_k, u_{k-1}x\}.$$

We omit the proof of this fact since it is similar to the proof of Claim 9. Note that the vertices x, u_1 and u_3 are completed.

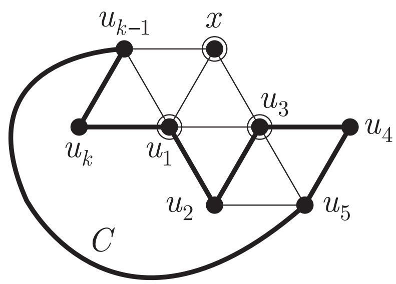

Fig. 12. Cycle C and graph $G(u_1, u_2, u_3, u_4, u_5, u_{k-1}, u_k, x)$

Starting from the construction of Fig. 12, we first show that $k \geqslant 9$ and specify the structure of graph G. Note that there exists a vertex u_6 in C such that $u_6 \not\sim \{u_2, u_4\}$. Indeed, if $u_2 \sim u_6$, then the cycle $xu_3u_4u_5u_2u_6Cu_1x$ is an extension of C. If $u_4 \sim u_6$, then the cycle $xu_3u_2u_5u_4u_6Cu_1x$ is an extension of C. Since $u_6 \not\sim \{u_2, u_4\}$, graph $G(N(u_5))$ is not isomorphic to P_4 and either $G(N(u_5)) \cong P_5$ or $G(N(u_5)) \cong C_6$. Let y be a neighbor of u_5 different from u_2, u_3, u_4 and u_6. Then either $y \sim \{u_2, u_6\}$, or $y \sim \{u_4, u_6\}$. By Claim 1, we have $y \in V(C)$, i.e., $y = u_i$, $7 \leqslant i \leqslant k-2$. Let us show that $i = 7$. Assuming the contrary, we have $8 \leqslant i \leqslant k-2$. If $u_i \sim \{u_4, u_6\}$, we arrive at the same contradiction as in the beginning of the proof of Claim 10. Let $u_i \sim \{u_2, u_6\}$. Then $u_{i+1} \not\sim \{u_2, u_6, u_{i-1}\}$ since otherwise C can be extended to the cycle $xu_3Cu_iu_2u_{i+1}Cu_1x$ if $u_{i+1} \sim u_2$, to the cycle $xu_3u_4u_5u_2u_i\overline{C}u_6u_{i+1}Cu_1x$ if $u_{i+1} \sim u_6$ and to the cycle $xu_3u_4u_5u_2u_iu_6Cu_{i-1}u_{i+1}Cu_1x$ if $u_{i+1} \sim u_{i-1}$. Hence, in $G(N(u_i))$ we have $u_{i+1} \not\sim \{u_2, u_6, u_{i-1}\}$, which is a contradiction to both $G(N(u_i)) \cong P_5$ and $G(N(u_i)) \cong C_6$. Thus, $i = 7$, $k \geqslant 9$, and either $u_2 \sim u_7$, or $u_4 \sim u_7$.

Suppose that $u_4 \sim u_7$. Let us show that vertices u_2 and u_5 are completed. Indeed, $G(N(u_5))$ cannot be isomorphic to C_6 since otherwise there exists a vertex z such that $z \sim \{u_2, u_6\}$. By Claim 1, $z \in V(C)$, i.e., $z = u_j$, $8 \leqslant j \leqslant k-2$. Moreover, $9 \leqslant j \leqslant k-2$ because of $G(N(u_5)) \cong C_6$. By considering $G(N(u_j))$, we can see that $u_{j+1} \not\sim \{u_2, u_6, u_{j-1}\}$ in exactly the same way as $u_{i+1} \not\sim \{u_2, u_6, u_{i-1}\}$ just shown. Hence, $G(N(u_j))$ is not isomorphic to P_5 or C_6, which is a contradiction to the condition of the theorem. Therefore, $G(N(u_5)) \cong P_5$ and the vertex u_5 is completed. As a consequence, u_2 is also completed and $G(N(u_2)) \cong P_3$. Indeed, its neighbors u_1, u_3 and u_5 are completed and, moreover, $u_2 \not\sim u_7$ (otherwise $G(N(u_5))$ is not isomorphic to P_5) and $u_2 \not\sim u_6$.

Suppose that $u_2 \sim u_7$. Now one can easily show that vertices u_4 and u_5 are completed. The proof is the same as in Claim 10 (case of $\deg u_2 = 5$).

Let $l = \lceil k/2 \rceil - 4$, where k is the length of C. Since $k \geqslant 9$, we have $l \geqslant 1$. To complete the proof, in l steps we arrive at a contradiction with the non-extendability of the cycle C. Let p denote the number of a step, $1 \leqslant p \leqslant l$. In the first step (when $p = 1$), the following is valid for both cases $u_2 \sim u_7$ and $u_4 \sim u_7$:

(a) there is an (x, u_7)-path in G with the vertex set $\{x, u_7\} \cup \{u_2, u_3, u_4, u_5\}$ (if $u_2 \sim u_7$, the path is $xu_3u_4u_5u_2u_7$, and if $u_4 \sim u_7$, the path is $xu_3u_2u_5u_4u_7$);
(b) there is a unique index r such that $r \in \{2p-2, 2p, 2p+2\}$ and $u_r \sim \{u_{2p+3}, u_{2p+5}\}$ (in particular, we have $r = 2$, $u_2 \sim \{u_5, u_7\}$ if $u_2 \sim u_7$, and $r = 4$, $u_4 \sim \{u_5, u_7\}$ if $u_4 \sim u_7$);
(c) all vertices $u_2, u_3, u_4, \ldots, u_{2p+3}$ are completed with the possible exception of u_r (in particular, from the sequence of completed vertices we have an exception of u_2 if $u_2 \sim u_7$, and an exception of u_4 if $u_4 \sim u_7$);
(d) $u_5 \sim u_7$, $\deg u_5 = 5$ and $\deg u_i \neq 6$, $2 \leqslant i \leqslant 5$.

Similarly to Claim 10, by induction on the number of steps p one can show that the following construction is valid in step $p+1$ if it is valid in step p for $1 \leqslant p < l$:

(a) there is an (x, u_{2p+5})-path Q_p in G with the vertex set $\{x, u_{2p+5}\} \cup \{u_2, u_3, \ldots, u_{2p+3}\}$;
(b) there is a unique index r such that $r \in \{2p-2, 2p, 2p+2\}$ and $u_r \sim \{u_{2p+3}, u_{2p+5}\}$;
(c) all vertices $u_2, u_3, u_4, \ldots, u_{2p+3}$ are completed with the possible exception of u_r;
(d) $u_{2p+3} \sim u_{2p+5}$, $\deg u_{2p+3} = 5$ and $\deg u_i \neq 6$, $2 \leqslant i \leqslant 2p+3$.

Obviously, this construction is valid in the first step. Supposing that it is valid in step p and considering the possible three cases for the index r, $r \in \{2p-2, 2p, 2p+2\}$, one can show that in all these cases the properties (a) – (d) are valid in step $p+1$ if they are valid in step p (we omit this proof which is similar to Claim 10).

The rest of the proof is also similar to the proof of Claim 10 and is reduced to considering two cases when k is even (i.e., $2l = k-8$ and $u_{2l+6} = u_{k-2}$) and k is odd (i.e., $2l = k-7$ and $u_{2l+6} = u_{k-1}$). In both cases, the assumption $\deg x = 3$ leads to a contradiction with the non-extendability of cycle C. This finishes the proof of the claim, the detailed verification being left to the reader (see Claim 10). □

Claim 13. *Relation* $\deg x \notin \{4, 5\}$ *holds, and* G *is isomorphic to graph* D *if* $\deg x = 6$.

Proof. Suppose that $\deg x \in \{4, 5, 6\}$. We know that $G(u_1, u_3, x)$ and $G(u_1, u_{k-1}, x)$ are triangles in G (from Claim 5) and $\deg u_1 = 5$ (from Claim 7). Since $G(N(u_1)) \cong P_5$, the vertex u_1 is completed. Let y be a neighbor of x different from u_1, u_3 and u_{k-1} such that either $y \sim u_3$, or $y \sim u_{k-1}$.

Consider only the case $y \sim u_3$ since to the case $y \sim u_{k-1}$ similar arguments apply. If $y \in S$, then by Claim 1, we have $y \not\sim u_4$. Note that $u_2 \not\sim u_4$ since otherwise the cycle $xu_3u_2u_4Cu_1x$ is an extension of C. In this case, $G(N(u_3))$ cannot be isomorphic to P_5 or C_6. Hence, $y \in V(C)$, i.e., $y = u_i$, $4 \leqslant i \leqslant k-2$. By Claim 1, $5 \leqslant i \leqslant k-3$. Let us show that $i = 5$. Assuming the contrary, we have $6 \leqslant i \leqslant k-3$. Since $u_2 \not\sim u_4$ and graph $G(N(u_3))$ is isomorphic either to P_5 or to C_6, we have $u_4 \sim u_i$. By Claim 1, we have $x \not\sim u_{i+1}$. Note that $u_{i+1} \not\sim \{u_4, u_{i-1}\}$ since otherwise C can be extended to the cycle $xu_i\overline{C}u_4u_{i+1}Cu_1u_2u_3x$ if $u_{i+1} \sim u_4$ and to the cycle $xu_iu_4Cu_{i-1}u_{i+1}Cu_1u_2u_3x$ if $u_{i+1} \sim u_{i-1}$. Hence, in $G(N(u_i))$ we have $u_{i+1} \not\sim \{u_4, u_{i-1}, x\}$, which is a contradiction to both $G(N(u_i)) \cong P_5$ and $G(N(u_i)) \cong C_6$. Thus, $i = 5$ and $u_3 \sim u_5$.

Now we show that vertices u_2 and u_3 are completed. Indeed, $G(N(u_3))$ cannot be isomorphic to C_6 since otherwise there exists a vertex z such that $z \sim \{u_2, u_4\}$. By Claim 1,

$z \in V(C)$, i.e., $z = u_j$, $7 \leqslant j \leqslant k-2$. Now we can easily see that $u_{j+1} \not\sim \{u_2, u_4, u_{j-1}\}$ since otherwise C can be extended to the cycle $xu_3Cu_ju_2u_{j+1}Cu_1x$ if $u_{j+1} \sim u_2$, to the cycle $xu_5Cu_ju_4u_{j+1}Cu_1u_2u_3x$ if $u_{j+1} \sim u_4$ and to the cycle $xu_3u_2u_ju_4Cu_{j-1}u_{j+1}Cu_1x$ if $u_{j+1} \sim u_{j-1}$. Hence, $G(N(u_j))$ is not isomorphic to P_5 or C_6, which is a contradiction to the condition of the theorem. Therefore, $G(N(u_3)) \cong P_5$ and the vertex u_3 is completed. As a consequence, u_2 is also completed and $G(N(u_2)) \cong P_2$. Indeed, its neighbors u_1 and u_3 are completed and, moreover, $u_2 \not\sim \{u_4, u_5, u_{k-1}, u_k, x\}$ since otherwise $G(N(u_1))$ or $G(N(u_3))$ is not isomorphic to P_5.

Let us show that the subgraph of G on the set of vertices $\{u_1, u_2, u_3, u_4, u_5, u_{k-1}, u_k, x\}$ is an induced subgraph, i.e., G has no edges u_4u_k, u_4u_{k-1}, u_5u_{k-1} and u_5u_k (see Fig. 13). If $u_4 \sim u_k$, then $xu_3u_2u_1u_ku_4Cu_{k-1}x$ is an extension of C. Suppose that $u_5 \sim u_k$. If $k = 7$, we have the extension $xu_1u_2u_3u_4u_5u_ku_{k-1}x$ of C. Hence, $k > 7$ and there exists a vertex u_6 in C different from u_{k-1}. If $u_k \sim u_6$, then $xu_1u_2u_3u_4u_5u_ku_6Cu_{k-1}x$ is an extension of C. Therefore, in $G(N(u_5))$ we have $u_k \not\sim \{x, u_4, u_6\}$, which is a contradiction to both $G(N(u_5)) \cong P_5$ and $G(N(u_5)) \cong C_6$. Hence, $u_5 \not\sim u_k$ and by symmetry we have $u_4 \not\sim u_{k-1}$. If $u_5 \sim u_{k-1}$, then we have a cycle on four vertices in $G(N(x))$ and arrive at a contradiction to the condition of the theorem.

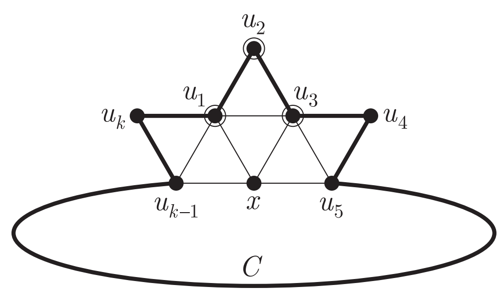

Fig. 13. Cycle C and graph $G(u_1, u_2, u_3, u_4, u_5, u_{k-1}, u_k, x)$

Now it is easily seen that Fig. 7 can be obtained from Fig. 13 by interchanging vertices x and u_2. Recall that in the proof of Claim 10 we arrived at a contradiction by constructing an extension of cycle C via vertex x (if $\deg x = 2$ and $\deg u_2 \in \{4, 5\}$). Consider now cycle $C' = u_1xu_3Cu_ku_1$ of the same length as cycle C. Then it is obvious that there exists an extension C'' of cycle C' via vertex u_2 if $\deg u_2 = 2$ and $\deg x \in \{4, 5\}$ (from the proof of Claim 10 and from the symmetry of the configurations in Fig. 7 and Fig. 13). Observe that cycle C'' is an extension of cycle C since $V(C) \subset V(C'')$ and $|V(C'')| = |V(C)| + 1$. Thus the assumption $\deg x \in \{4, 5\}$ leads to a contradiction with the non-extendability of C, and we have the only possibility $\deg x = 6$.

Furthermore, we have $\deg x = 6$, $\deg u_2 = 2$ and from the symmetry of the configurations in Fig. 7 and Fig. 13 we arrive to the conditions of Claim 11. Hence we can repeat the arguments of the proof of Claim 11 (with extension of cycle C' via vertex u_2 instead of extension of cycle C via vertex x) and conclude that G is isomorphic to graph D.

This finishes the proof of the claim. □

Summarizing the above claims, we conclude that G is either isomorphic to graph D, or fully cycle extendable. This completes the proof of the theorem. □

Next we give immediate consequences of Theorems 1, 2 and Observation 3.

Corollary 1. *Let G be a connected, locally connected triangular grid graph. Then G is either fully cycle extendable or isomorphic to the graph D.*

Corollary 2. *Let G be a 2-connected, linearly convex triangular grid graph. Then G is either fully cycle extendable or isomorphic to the graph D.*

Thus, the main result of [17] on the hamiltonicity of 2-connected, linearly convex triangular grid graphs directly follows from Corollary 2.

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