COMPLEXITY OF THE HAMILTONIAN CYCLE PROBLEM IN TRIANGULAR GRID GRAPHS

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ABSTRACT. A triangular grid graph is a finite induced subgraph of the infinite graph associated with the two-dimensional triangular grid. We show that the problem HAMILTONIAN CYCLE is NP-complete for triangular grid graphs, while a hamiltonian cycle in connected, locally connected triangular grid graph can be found in polynomial time.
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1. **Introduction**

In this paper, we prove that the HAMILTONIAN CYCLE problem is NP-complete for triangular grid graphs, while a hamiltonian cycle in connected, locally connected triangular grid graph can be found in polynomial time. A triangular grid graph is a finite induced subgraph of the infinite graph associated with the two-dimensional triangular grid. Some hamiltonian properties of triangular grid graphs are considered in [10, 11]. Such properties are important in applications connected with problems arising in molecular biology (protein folding) [1], in configurational statistics of polymers [3, 9], in telecommunications and computer vision (problems of determining the shape of an object represented by a cluster of points on a grid). Cyclic properties of triangular grid graphs can also be used in the design of cellular networks since these networks are generally modelled as induced subgraphs of the infinite two-dimensional triangular grid [6].

For graph-theoretic terminology not defined in this paper, the reader is referred to [2]. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For each vertex u of G, the *neighborhood* $N(u)$ of u is the set of all vertices adjacent to u. The *degree* of u is defined as $\deg u = |N(u)|$. For a subset of vertices $X \subseteq V(G)$, the subgraph of G induced by X is denoted by $G(X)$. A vertex u of G is said to be *locally connected* if $G(N(u))$ is connected. G is called *locally connected* if each vertex of G is locally connected.

We say that G is *hamiltonian* if G has a *hamiltonian cycle*, i.e., a cycle containing all vertices of G. A path with the end vertices u and v is called a (u, v)-*path*. A (u, v)-path is a *hamiltonian path* of G if it contains all vertices of G. As usual, P_k and C_k denote the path and the cycle on k vertices, respectively. In particular, C_3 is a *triangle*. The path

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P (respectively, cycle C) on k vertices $v_1, v_2, \ldots, v_k$ with the edges v_iv_{i+1} (respectively, v_iv_{i+1} and v_1v_k) $(1 \leqslant i < k)$ is denoted by $P = v_1v_2 \ldots v_k$ (respectively, $C = v_1v_2 \ldots v_kv_1$).

A cycle C in a graph G is *extendable* if there exists a cycle C' in G (called the *extension* of C) such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. If such a cycle C' exists, we say that C *can be extended to* C'. If every non-hamiltonian cycle C in G is extendable, then G is said to be *cycle extendable*. We say that G is *fully cycle extendable* if G is cycle extendable and each of its vertices is on a triangle of G. Clearly, any fully cycle extendable graph is hamiltonian.

The infinite graph T^∞ associated with the two-dimensional triangular grid (also known as *triangular tiling graph* [5, 14]) is a graph drawn in the plane with straight-line edges and defined as follows. The vertices of T^∞ are represented by a linear combination $x\boldsymbol{p} + y\boldsymbol{q}$ of the two vectors $\boldsymbol{p} = (1, 0)$ and $\boldsymbol{q} = \left(1/2, \sqrt{3}/2\right)$ with integers x and y. Thus we may identify the vertices of T^∞ with pairs (x, y) of integers, and thereby the vertices of T^∞ are points with Cartesian coordinates $\left(x + y/2, y\sqrt{3}/2\right)$. Two vertices of T^∞ are adjacent if and only if the Euclidean distance between them is equal to 1 (see Fig. 1). Note that the degree of any vertex of T^∞ is equal to six. A *triangular grid graph* is a finite induced subgraph of T^∞. A triangular grid graph G is *linearly convex* if, for every line l which contains an edge of T^∞, the intersection of l and G is either a line segment (a path in G), or a point (a vertex in G), or empty. For example, the triangular grid graph G (with three components including an isolated vertex w) shown in Fig. 2 is linearly convex even though G has vertices u and v whose midpoint z is a vertex of T^∞ but not of G. In Fig. 2, dark points correspond to the vertices of T^∞.

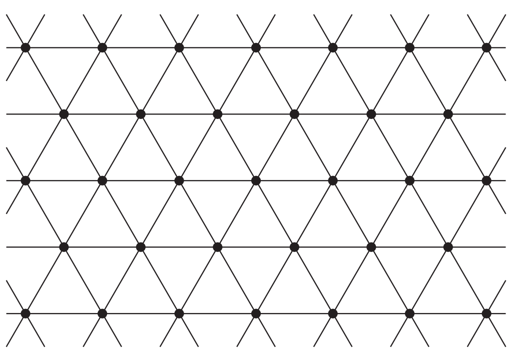

Fig. 1. A fragment of graph T^∞

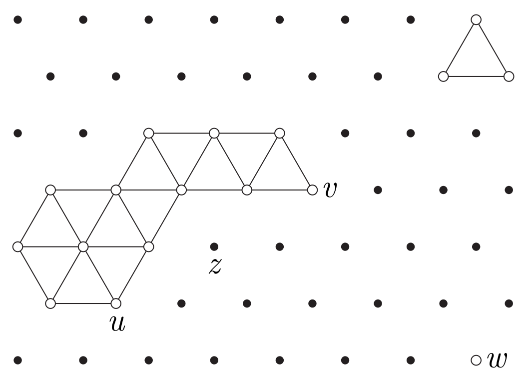

Fig. 2. An example of triangular grid graph

As has been shown by Reay and Zamfirescu [14], all 2-connected, linearly convex triangular grid graphs (or T-graphs in the terminology of [14]) are hamiltonian (with the exception of one of them). The only exception is a graph D which is the linearly-convex hull of the Star of David; this graph is 2-connected and linearly convex but not hamiltonian (see Fig. 3). We extend this result to a wider class of locally connected triangular grid graphs in [11].

In the following section, the Hamiltonian Cycle problem is shown to be NP-complete for triangular grid graphs. The results of this paper were briefly announced in [10, 12].

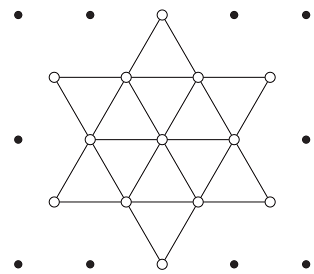

Fig. 3. Graph D

2. NP-completeness of the problem Hamiltonian Cycle for triangular grid graphs

Consider the following well-known decision problem.

Hamiltonian Cycle
Instance: A graph G.
Question: Is G hamiltonian?

The complexity of the problem Hamiltonian Cycle has been intensively investigated. The problem is NP-complete for general graphs and remains difficult for graphs of many special classes [4]. Among them, there are bipartite graphs, line graphs, 3-connected cubic (i.e., 3-regular) planar graphs, maximal planar graphs and others. By Itai et al. [7], it has been proved that the Hamiltonian Cycle problem is NP-complete for grid graphs. We use the idea of this proof for showing that the problem remains NP-complete for triangular grid graphs. Notice that grid graphs are not a subclass of triangular grid graphs: these classes of graphs have common elements but in general they are distinct.

The interrelation between grid graphs and triangular grid graphs is as follows. A *grid graph* is a finite induced subgraph of the infinite graph G^∞ associated with the two-dimensional rectangular grid, i.e., G^∞ is the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if and only if the Euclidean distance between them is equal to 1.

Let us introduce graph S^∞ obtained from graph T^∞ by deleting all edges on the lines traced from up-left to down-right (see the dashed lines in Fig. 4). Note that graph S^∞ is isomorphic to G^∞ but these graphs are different when considered as geometric graphs. Let a *slope grid graph* be a finite induced subgraph of S^∞. Introduce a slope graph as follows. Recall that the vertices of T^∞ are identified with pairs (x, y) of integers, and each vertex (x, y) has six neighbors $(x \pm 1, y)$, $(x, y \pm 1)$, $(x + 1, y - 1)$ and $(x - 1, y + 1)$. Similarly, the vertices of S^∞ can be identified with the same pairs of integers and each vertex (x, y)

has four neighbors $(x \pm 1, y)$, $(x, y \pm 1)$. A *slope graph* $S(m, n)$ is a slope grid graph whose vertex set is $\{(x+i, y+j) \mid 0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant n-1\}$. Thus the integers m and n specify a slope graph up to isomorphism. Note that the slope graph $S(m, n)$ is isomorphic to the rectangular graph $R(m, n)$ introduced in the proof of Itai et al. [7].

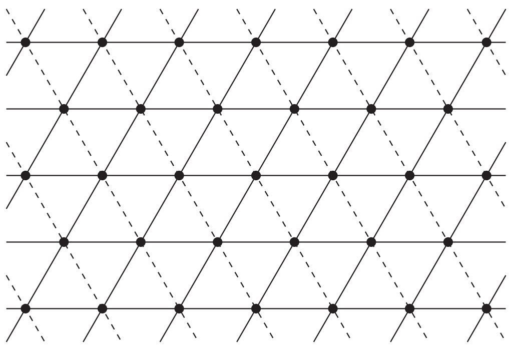

Fig. 4. A fragment of graph T^∞ with dashed lines

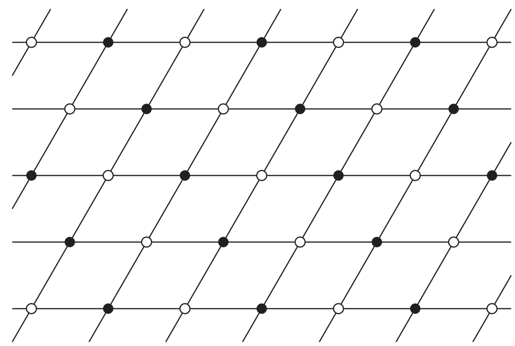

Fig. 5. A fragment of graph S^∞ with marked vertices

Note that both graphs G^∞ and S^∞ are bipartite. Similarly to even and odd vertices of G^∞ in the proof of Itai et al. [7], we introduce white and black vertices of S^∞ in the following way. Starting from an arbitrary vertex v of S^∞ mark it as a white one; all vertices with an even distance from v mark as white, and all other vertices mark as black (see Fig. 5).

Theorem 1. *The problem* Hamiltonian Cycle *is NP-complete for triangular grid graphs.*

Proof. Clearly, the problem is in NP. To prove that it is NP-complete, we establish a polynomial-time reduction from the Hamiltonian Cycle problem for planar cubic bipartite graphs which is shown to be NP-complete by Plesnik [13].

Let $B = (V^0, V^1, E)$ be a planar cubic bipartite graph and G_1 be a slope graph. Similarly to the parity-preserving embedding [7] of a bipartite graph into a rectangular graph, let us introduce the following parity-preserving embedding *emb* of B into G_1 (a one-to-one function from $V^0 \cup V^1$ to the vertices of G_1 and from E to paths in G_1):

1. The vertices of V^0 are mapped to white vertices of G_1, i.e., $emb(u)$ is white if $u \in V^0$.
2. The vertices of V^1 are mapped to black vertices of G_1, i.e., $emb(u)$ is black if $u \in V^1$.
3. The edges of B are mapped to vertex-disjoint (except perhaps for end vertices) paths of G_1, i.e., if $uv \in E$, then $emb(uv)$ is a path P from $emb(u)$ to $emb(v)$, and the internal vertices of P do not belong to any other path.

See Fig. 6 for an example of a parity-preserving embedding of the planar cubic bipartite graph B on eight vertices (Fig. 6a) into a slope graph. The resulting graph $emb(B)$ is shown in Fig. 6b.

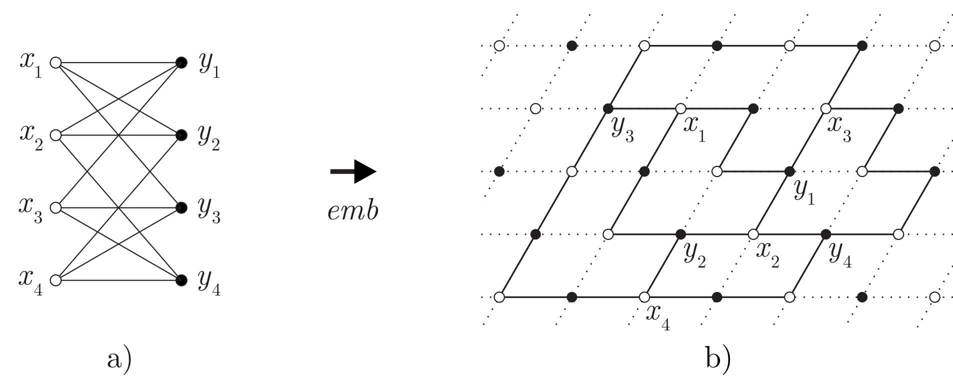

Fig. 6. An example of a parity-preserving embedding

As shown in [7], one can construct in polynomial time a parity-preserving embedding of a planar cubic bipartite graph B with n vertices into a rectangular graph $R(kn, kn)$ (for some constant k). Similarly, our parity preserving embedding of B into a slope graph $S(kn, kn)$ for some constant k can also be constructed in polynomial time.

Now given a planar cubic bipartite graph B with n vertices, we shall construct a triangular grid graph G_7 such that G_7 has a hamiltonian cycle if and only if there exists a hamiltonian cycle in B.

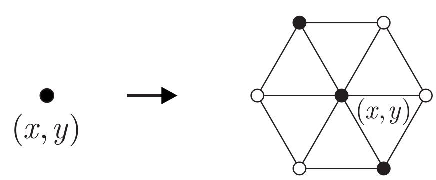

Fig. 7. A 7-cluster of G_7

First we embed (as described above) graph B into a slope graph $G_1 = S(kn, kn)$ for some constant k. To obtain graph G_7, in the first step we construct a slope graph G'_1 by multiplying the scale of G_1 by 7, i.e., each edge of G_1 is transformed into a path with 7 edges. Let vertex (x, y) be the image of a vertex of B in G'_1, and the color (white or black) of the image be the same as in G_1. In the second step, we transform the slope graph G'_1 into a triangular grid graph by adding edges from up-left to down-right (using the transformation inverse to the transformation of T^∞ into S^∞) and by inheriting the colors of the vertices. In the third step, the image (x, y) of each vertex of B is transformed into the 7-cluster of G_7 (as shown in Fig. 7 for the case of the black vertex (x, y)). Each 7-cluster is a wheel W_6 with either white or black central vertex. Finally, in the fourth step the edges of B are simulated by tentacles in G_7 as described below (this process is explained by Fig. 8–12).

Before giving the tentacle definition, we have to determine a strip.

A *strip* is a triangular grid graph which is isomorphic to the square P_k^2 of the path P_k for some $k \geqslant 4$. Remind that the square G^2 of graph G is a graph on $V(G)$ in which two vertices are adjacent if and only if they have a distance at most 2 in G. There are three possible orientations of strips on four vertices and six possible orientations of strips on $k \geqslant 5$ vertices. Examples of all possible orientations of strips on five and six vertices are shown in Fig. 8a and Fig. 8b. Notice that each strip has two terminal triangles and strips on $k \geqslant 5$ vertices have also internal triangles.

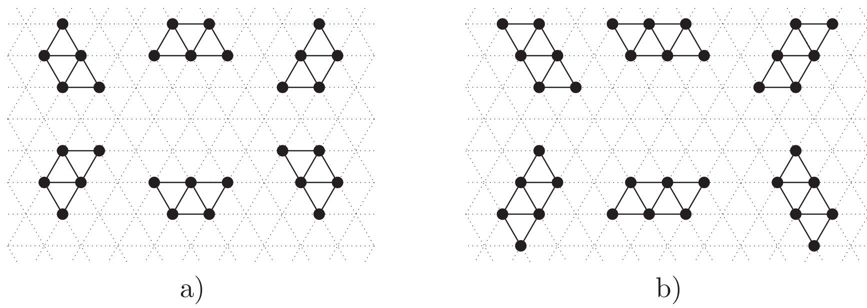

Fig. 8. Strips on five and six vertices

A *tentacle* is a triangular grid graph which is either a strip or a series of strips stuck together by the edges of terminal triangles as shown in Fig. 9b. The stuck together edges of terminal triangles are given thick in Fig. 9b; the end vertices of these edges form *inner corners* of the tentacle. There are four *outer corner* vertices in each tentacle. These vertices have a degree either 2 or 3, all other vertices of the tentacle have degree 4 (except inner corners which have degrees 3 or 5).

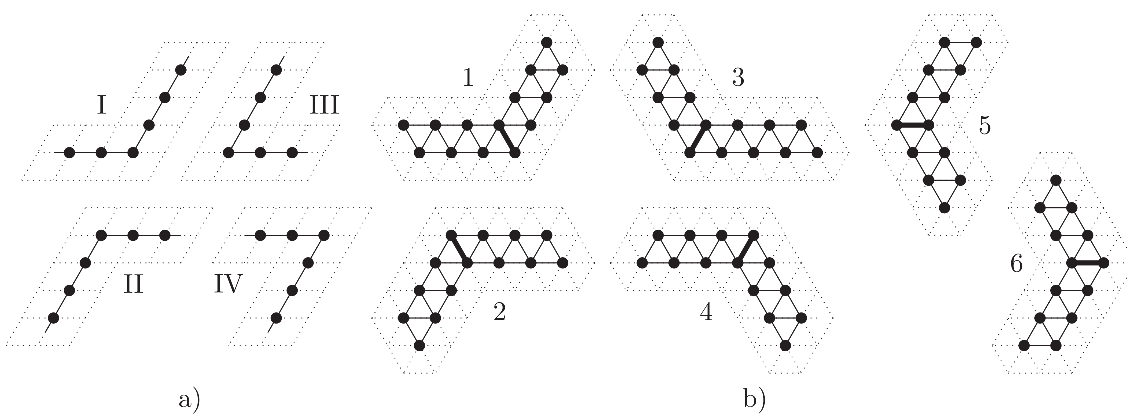

Fig. 9. Feasible sticking of strips to form a tentacle

Note that Fig. 9b shows all feasible ways of sticking strips to form tentacles (and any other sticking is unfeasible) since the parity-preserving embedding gives only four possible types of the turns of paths in the resulting graph $emb(B)$ (see Fig. 9a). For instance, in the parity-preserving embedding shown in Fig. 6, the path P_1 from x_4 to y_4 has a turn of type I, the path P_2 from x_3 to y_3 has turns of types IV and II, the path P_3 from x_4 to y_3 has a turn of type III. Types I and II of the turns (Fig. 9a) lead to the cases 1 and 2 of sticking, respectively (Fig. 9b). If the turn of the path is of type III, then the corresponding tentacle is constructed as a consequence of sticking in case 5 and then in case 3 (see Fig. 10a). If the turn of the path is of type IV, then the tentacle is constructed as a consequence of sticking in case 4 and then in case 6 (see Fig. 10b).

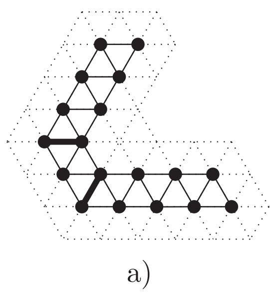

Fig. 10. Tentacles for the types III and IV of the turns of the paths

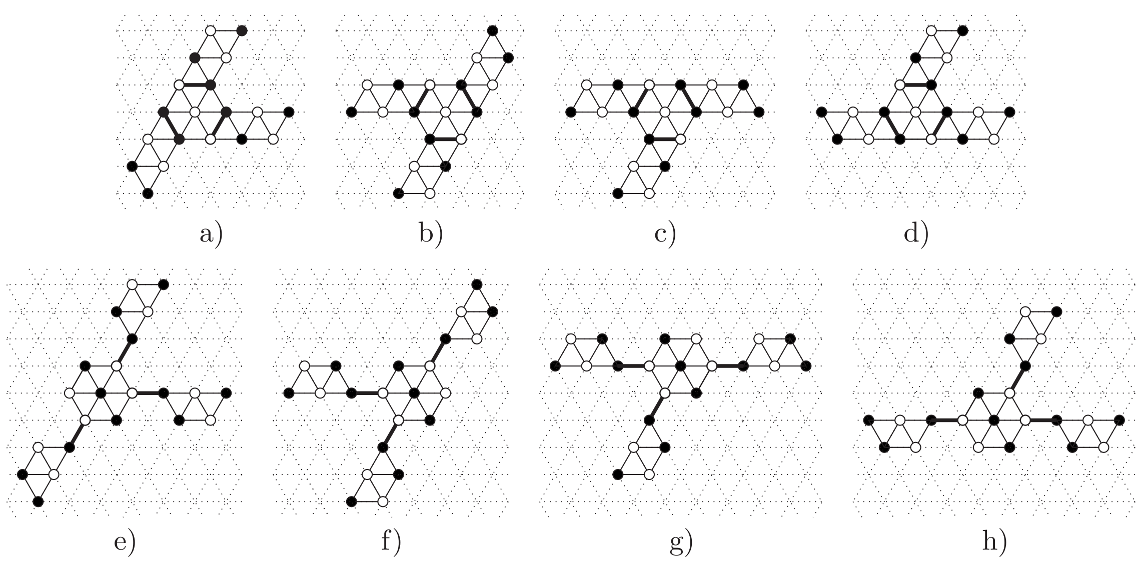

Fig. 11. Possible types of connection of the 7-cluster to the tentacle

The edges of B are simulated by tentacles in the following way. Let uv be an edge from $u \in V^0$ to $v \in V^1$. Consider the path in $emb(B)$ corresponding to uv. Graph G_7 will include the blown up image of this path as a tentacle connected to the 7-clusters corresponding to u and v. The possible types of connection of the tentacles to the 7-cluster are shown in Fig. 11, where cases (a–d) correspond to a white vertex of graph B and cases (e–h) correspond to a black vertex of B. The places of connection by edges are given thick in Fig. 11.

Let the edge of B correspond to the path of $emb(B)$ that has no turns. Then this edge can be simulated as a tentacle which is either a strip or consists of three strips (see Fig. 12, where $emb(B)$ has two paths P_1 and P_2 without turns: edge uv of B corresponds to (u, v)-path P_1 and edge vw of B corresponds to (v, w)-path P_2; these paths and the corresponding tentacles of G_7 are given thick). Thus the number of vertices in any tentacle in G_7 is at least 10.

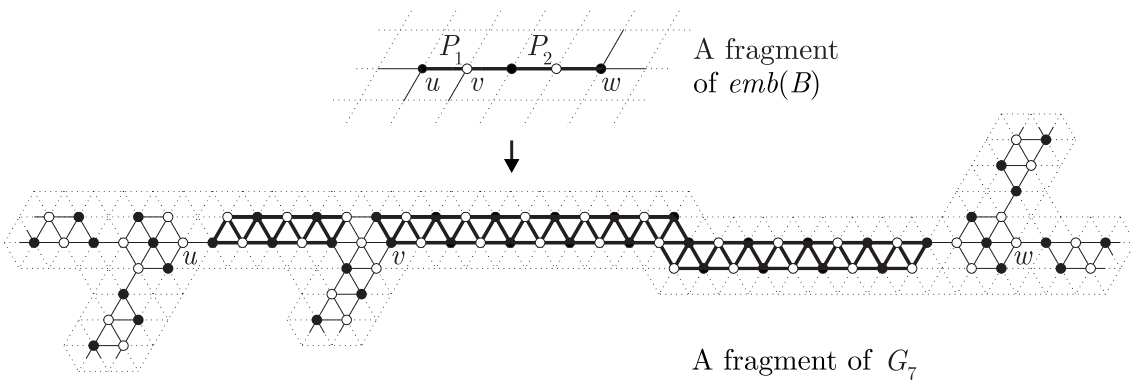

Fig. 12. An example of tentacles for the paths without turns

This concludes the description of graph G_7. It is clear that graph G_7 is constructed from graph B in polynomial time (with respect to the number of vertices and edges of B). An example of G_7 corresponding to the graph B in Fig. 6a is shown in Fig. 13.

Let u and v be any outer corner vertices of a tentacle T in G_7. It is easy to see that there are only two types of hamiltonian (u, v)-paths in T. The path can be either a *return* path if u and v are adjacent or a *cross* path if u and v are not adjacent (see Fig. 14 with a fragment of graph G_7 given in Fig. 13, where Fig. 14a shows the return path and Fig. 14b,c show some of cross paths given thick). It is evident that in the former case, vertices u and v are in the same terminal triangle of T and in the latter case, u and v are in different terminal triangles. Notice that there is only one return path in a tentacle in contrast to a set of cross paths.

Moreover, if $\deg_T v = 2$ (the degree of vertex v in T) and the outer corner vertices u and v are not adjacent, then independent of $\deg_T u$ there exists a cross (u, v)-path in T. This can be shown by an easy induction on the number of strips.

The following claim completes the proof of the theorem.

Claim 1. *Graph G_7 has a hamiltonian cycle if and only if there exists a hamiltonian cycle in graph B.*

Proof. Let graph B have a hamiltonian cycle C. We construct the corresponding hamiltonian cycle HC_7 in G_7 as follows. Let uv be an edge of graph B and uv be simulated by

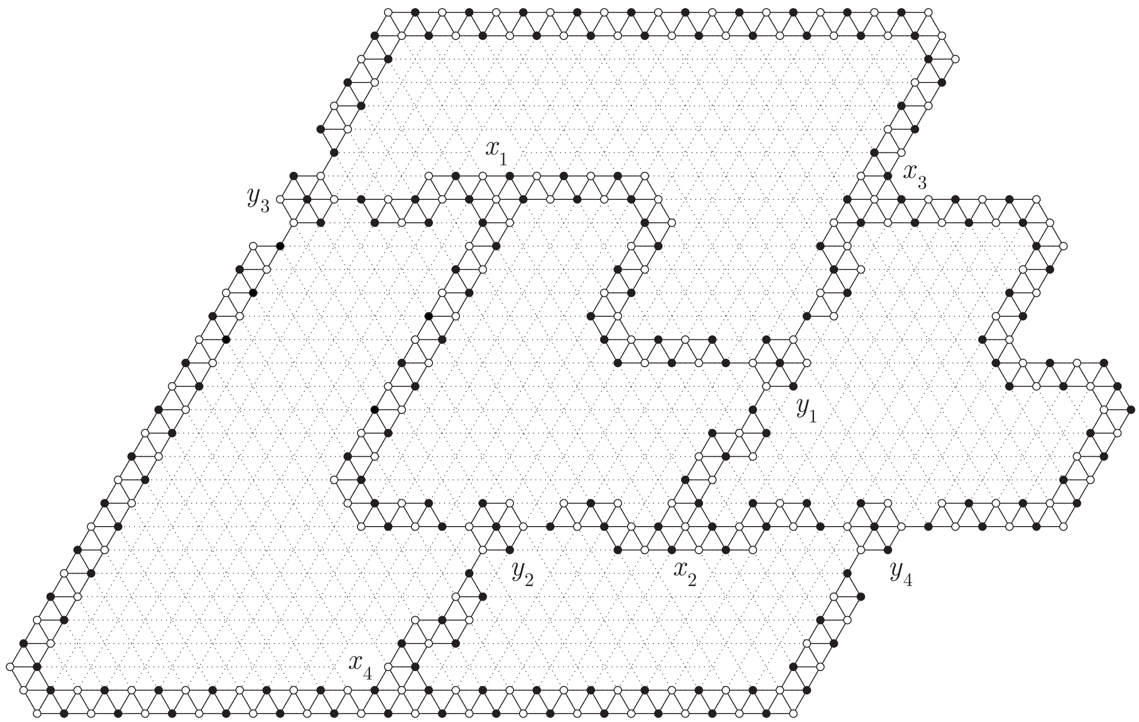

Fig. 13. Graph G_7

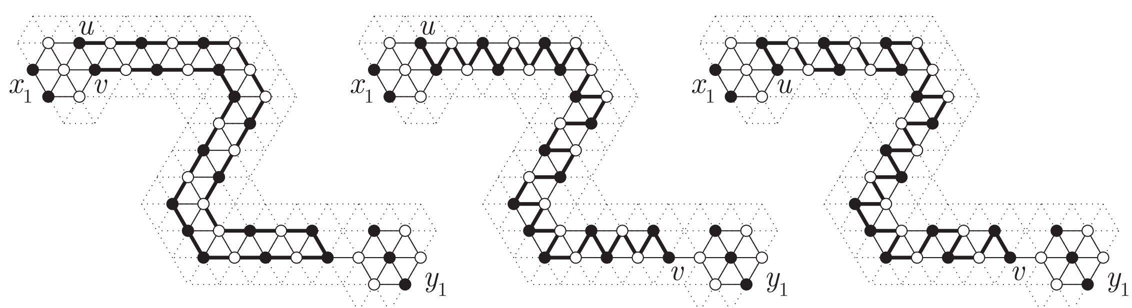

Fig. 14. Return (a) and cross (b and c) paths of a tentacle

a tentacle T_{uv} in G_7. Starting to form HC_7, we will cover T_{uv} by a cross path if uv is in C, and by a return path otherwise.

Moreover, any return path is constructed in such a way that its end vertices belong to a cluster with a white central vertex. Denote by C_w a cluster with a white central vertex, by c the central vertex of this cluster, and by x_1, x_2 the end vertices of the return path. Denote by T_1 and T_2 two other tentacles connected to C_w. Let a_i, $i = 1, 2$, be a vertex of $V(C_w) \cap V(T_i)$ such that $a_i \sim x_i$, and let b_i be an outer corner vertex of T_i such that

$b_i \in V(T_i) \setminus V(C_w)$ and $\deg_{T_i} b_i = 2$. The cross path for T_i, $i = 1, 2$, is constructed in such a way that it starts from the vertex a_i and ends in the vertex b_i. Note that each of the vertices b_1 and b_2 is adjacent to a vertex of different clusters with a black central vertex.

The partial paths can be connected to constitute HC_7 by covering the clusters in the following way. To cover cluster C_w we add the edges a_1x_1, a_2c and cx_2. Each cluster C_b with a black central vertex is connected with three tentacles, two of them, say T_3 and T_4, covered by cross paths. Let b_i, $i = 3, 4$, be the end vertex of the cross path of T_i such that b_i is adjacent to a vertex of C_b. Cluster C_b is covered by a path from b_3 to b_4 through all vertices of C_b. This completes the construction of HC_7.

Assume now that graph G_7 has a hamiltonian cycle HC_7. Since there are only two types of hamiltonian (u, v)-paths in a tentacle with outer corner vertices u and v, each tentacle is covered either by a cross path or by a return path which are partial paths of HC_7. To construct a hamiltonian cycle C in graph B, we include in C all edges corresponding to tentacles covered by cross paths. Note that C can be considered as obtained from HC_7 by a contraction of each 7-cluster into a vertex of B and by a transformation of each cross path into an edge of B. This is a hamiltonian cycle because each 7-cluster cannot be covered by HC_7 unless it is incident upon exactly two cross paths. □

This completes the proof of the theorem. □

Note that Corollary 1 from [11] implies the polynomial solvability of the Hamiltonian Cycle problem for locally connected triangular grid graphs. Moreover, from the proof of Theorem 2 in [11] it can be shown that the following statement holds.

Theorem 2. *Let G be a connected graph not isomorphic to D and for any vertex u of G the subgraph $G(N(u))$ be isomorphic to one of the graphs P_2, P_3, P_4, P_5, or C_6. If C is a cycle of length k in G and $3 \leqslant k < |V(G)|$, then a cycle C' of length $k+1$ such that $V(C) \subset V(C')$ can be found in polynomial time.*

The following corollary is an immediate consequence of Theorem 2.

Corollary 1. *A hamiltonian cycle in a connected, locally connected triangular grid graph (not isomorphic to D) can be found in polynomial time.*

An example of a locally connected triangular grid graph with one of its hamiltonian cycles (bold lined) is shown in Fig. 15. This graph is not linearly convex and contains holes. Note that a polynomial algorithm for finding a hamiltonian cycle in a grid graph is known only in the case when the graph does not contain holes [8].

Finally consider the following problem.

Hamiltonian (u, v)-Path
Instance: A graph G and vertices $u, v \in V(G)$.
Question: Does G contain a hamiltonian (u, v)-path?

The following statement can be proved using Theorem 3.

Theorem 3. *The problem* Hamiltonian (u, v)-Path *is NP-complete for triangular grid graphs.*

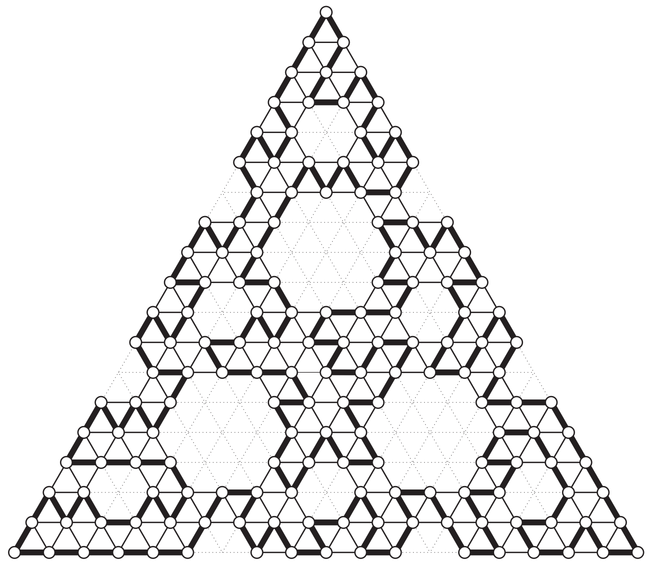

Fig. 15. A locally connected triangular grid graph and one of its hamiltonian cycles

The scheme of the proof is the following. Given a planar cubic bipartite graph B, we construct a triangular grid graph G_7 in the same way as in Theorem 1. We transform graph G_7 into a triangular grid graph G'_7 as follows. For a cluster C_b with a black central vertex we add two new vertices u and v adjacent to vertices $x, y \in V(C_b)$, where $x \sim y$ and $\deg_{G_7} x = \deg_{G_7} y = 3$. These new vertices have degree 1 in graph G'_7, and there exists a hamiltonian (u, v)-path in G'_7 if and only if graph G_7 has a hamiltonian cycle. Thus Claim 1 and NP-completeness of the Hamiltonian Cycle problem for planar cubic bipartite graphs complete the scheme of the proof of Theorem 3.

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