GRAPHS WITH MAXIMAL INDUCED MATCHINGS OF THE SAME SIZE

PHILIPPE BAPTISTE, MIKHAIL Y. KOVALYOV, YURY L. ORLOVICH, FRANK WERNER, IGOR E. ZVEROVICH

ABSTRACT. A graph is well-indumatched if all its maximal (with respect to set inclusion) induced matchings are of the same size. We first prove that recognizing the class $\text{WIM}$ of well-indumatched graphs is a co-NP-complete problem even for $(2P_5, K_{1,5})$-free graphs. We then show that the well-known decision problems such as INDEPENDENT DOMINATING SET, INDEPENDENT SET, and DOMINATING SET are NP-complete for well-indumatched graphs. We also show that $\text{WIM}$ is a co-indumatching hereditary class and characterize well-indumatched graphs in terms of forbidden co-indumatching subgraphs. However, we prove that recognizing co-indumatching subgraphs is an NP-complete problem. A graph $G$ is perfectly well-indumatched if every induced subgraph of $G$ is well-indumatched. We characterize the class of perfectly well-indumatched graphs in terms of forbidden induced subgraphs. Finally, we show that both INDEPENDENT DOMINATING SET and INDEPENDENT SET can be solved in polynomial time for perfectly well-indumatched graphs, even in their weighted versions, but DOMINATING SET is still NP-complete.

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1. Introduction

In this paper, we study the recognition problem of graphs, where all maximal (with respect to set inclusion) induced matchings have the same number of edges, and examine the computational complexity of certain fundamental graph problems such as maximum independent set, minimum dominating set, and minimum independent dominating set within these graphs.

An induced matching $M$ of a graph $G$ is a set of pairwise non-adjacent edges such that no two edges of $M$ are joined by an edge in $G$. An induced matching $M$ is maximal if no other induced matching in $G$ contains $M$. Induced matchings have applications in the areas of communication network testing [26], concurrent transmission of messages in wireless ad hoc networks [1], secure communication channels in broadcast networks [16], and many others.

A graph $G$ is called well-indumatched if all maximal induced matchings in $G$ have the same size. For example, the graph obtained from a star $K_{1,n}$ by subdividing each edge...
of $K_{1,n}$ by two vertices is a well-indumatched graph. We denote by $WIM$ the class of well-indumatched graphs. An interesting property of a well-indumatched graph is that its maximal induced matching is a maximum induced matching. Therefore, the class $WIM$ forms the set of greedy instances with respect to the maximum induced matching problem, because it is solved for well-indumatched graphs by any greedy type algorithm. Greedy instances of other combinatorial problems can be defined in a similar way ([6], see also [29]).

Caro et al. [6] investigated the complexity of recognizing greedy instances of several combinatorial problems, one of which is the maximum independent set problem. The class of well-covered graphs is a class of greedy instances of the maximum independent set problem and has applications, in particular, in distributed computing systems [30].

The concept of a well-covered graph (with all its maximal independent sets of the same size) was first introduced by Plummer [23] and has been investigated extensively in the literature. Ravindra [24] characterized the well-covered trees and bipartite graphs. Finbow et al. [10] characterized the well-covered graphs with a girth of at least 5. Then, Campbell et al. [5] characterized the well-covered cubic graphs. In a series of papers, Finbow et al. [11, 12, 13] characterized the well-covered plane triangulations. The problem of recognizing the well-covered graphs was shown to be co-NP-complete for general graphs, independently by Sankaranarayana and Stewart [25], and Chvátal and Slater [8]. It is co-NP-complete even for $K_{1,4}$-free graphs [6], and it is solvable in polynomial time for $K_{1,3}$-free graphs [27, 28].

A matching of a graph $G$ is a set of edges in $G$ with no common end-vertices. The problem of graphs recognition, in which all maximal matchings have the same size, was first considered by Lesk et al. [21]. Graphs which satisfy this property are known in the literature as equimatchable. In [21], it is shown that there exists a polynomial-time algorithm which decides whether a given input graph is equimatchable (see also Lovász and Plummer [22] for a detailed description of equimatchable graphs).

The class $WIM$ of well-indumatched graphs is a natural analogue of well-covered graphs. We show that recognizing the class $WIM$ is a co-NP-complete problem even for $(2P_5, K_{1,5})$-free graphs. Thus, it is unlikely that there exists a characterization of well-indumatched graphs which provides its polynomial recognition.

Let IMatch($G$) be the set of all maximal induced matchings of graph $G$. Define the minimum maximal induced matching number as

$$\sigma(G) = \min\{|M| : M \in \text{IMatch}(G)\}$$

and the maximum induced matching number as

$$\Sigma(G) = \max\{|M| : M \in \text{IMatch}(G)\}.$$  

In a greedy way we can find both $\sigma(G)$ and $\Sigma(G)$ in any well-indumatched graph $G$. It is well known that the decision analogue of the problem of computing $\Sigma(G)$ is NP-complete [4, 26]. We prove NP-completeness for the problem with $\sigma(G)$ even if graphs have maximal induced matchings of at most two sizes.

Ko and Shepherd [19] investigated relations between $\Sigma(G)$ and $\gamma(G)$, the domination number of $G$. They mentioned that they do not know any class of graphs for which
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exactly one of $\gamma$, $\Sigma$ is polynomial-time computable. We show that \textsc{Independent Dominating Set}, \textsc{Independent Set}, and \textsc{Dominating Set} are NP-complete problems for well-indumatched graphs. Thus, for the class $\mathcal{G}M$, $\gamma$ is hard to find, while $\Sigma$ is easily computable. Our construction for the \textsc{Independent Set} problem implies that the well-known problems \textsc{Partition into Triangles} and \textsc{Chordal Graph Completion} are NP-complete for well-indumatched graphs. Furthermore, \textsc{Partition into Subgraphs Isomorphic to $P_3$} is an NP-complete problem for well-indumatched graphs. This implies that computing $\Sigma$ is NP-hard even if the input is restricted to Hamiltonian line graphs of well-indumatched graphs, which generalizes results of Kobler and Rotics [20]. Also, \textsc{Graph $k$-Colorability} and \textsc{Clique} are NP-complete problems for well-indumatched graphs.

A class of graphs is called \textit{hereditary} if every induced subgraph of a graph in this class also belongs to the class. For a set $\mathcal{H}$ of graphs, a graph $G$ is called $\mathcal{H}$-\textit{free} if no induced subgraph of $G$ is isomorphic to a graph in $\mathcal{H}$. In other words, $\mathcal{H}$-free graphs constitute a hereditary class defined by $\mathcal{H}$ as the set of forbidden induced subgraphs.

We show that $\mathcal{G}M$ is a co-indumatching hereditary class, i.e., it is closed under deleting the end-vertices of an induced matching along with their neighborhoods. We characterize well-indumatched graphs in terms of forbidden co-indumatching subgraphs. It means that we specify the minimal set of graphs $\mathcal{X}$ such that $G$ is well-indumatched if and only if $G$ does not contain any graph in $\mathcal{X}$ as a co-indumatching subgraph. Unfortunately, recognizing co-indumatching subgraphs is NP-complete.

Finally, we consider \textit{perfectly well-indumatched} graphs, i.e., graphs in which every induced subgraph is well-indumatched. We characterize the class of perfectly well-indumatched graphs in terms of forbidden induced subgraphs, thus obtaining a new polynomial-time recognizable class, where both parameters $\sigma$ and $\Sigma$ are easy to compute. We show that both \textsc{Independent Set} and \textsc{Independent Dominating Set} can be solved in polynomial time for perfectly well-indumatched graphs, even in their weighted versions, but \textsc{Dominating Set} is NP-complete.

In this paper, we use standard graph-theoretic terminology of Bondy and Murty [3]. Let $G$ be a graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The subgraph of $G$ induced by a set $X \subseteq V$ is denoted by $G[X]$, and $G - X = G(V \setminus X)$. $N_G(x)$ is the neighborhood and $N_G[x] = \{x\} \cup N_G(x)$ is the closed neighborhood of a vertex $x \in V(G)$. Also, for a set $X \subseteq V(G)$, $N_G(X) = \cup_{x \in X} N_G(x)$, $N_G[X] = X \cup N_G(X)$, and $E_G(X)$ is the set of all edges in $G$ that have at least one end-vertex in $X$. The complete graph, the chordless path and the chordless cycle on $n$ vertices are denoted by $K_n$, $P_n$ and $C_n$, respectively. Let $K_0$ be the \textit{null graph}, i.e., $K_0$ has no vertices and edges. $K_4 - e$ is a graph obtained from the complete graph $K_4$ by deleting an edge. The \textit{star} $K_{1,n}$ consists of a dominating vertex and $n$ pendant vertices. The star $K_{1,3}$ is also known as the \textit{claw}.

We use the notation $nG$ for the disjoint union of $n$ copies of $G$. We denote by $G^2$ the \textit{square} of graph $G$, i.e., the graph on $V(G)$ in which two vertices are adjacent if and only if they have a distance of at most 2 in $G$. Finally, $L(G)$ is the \textit{line graph} of a graph $G$, i.e., $V(L(G)) = E(G)$, and two vertices $e$ and $e'$ are adjacent in $L(G)$ if and only if the edges $e$ and $e'$ are adjacent in $G$. 

2. Complexity of recognizing well-indumatched graphs

We consider the following decision problem.

**Non-Well-Indumatched Graphs**

**Instance:** A graph $G$.

**Question:** Are there two maximal induced matchings $M$ and $N$ in $G$ with $|M| \neq |N|$?

We first prove that **Non-Well-Indumatched Graphs** is an NP-complete problem. Then we extend this result by showing that the recognition of graphs having maximal induced matchings of at most $t$ sizes is co-NP-complete for any value of $t \geq 1$.

For the proof of NP-completeness, we will use a polynomial-time reduction from the following well-known NP-complete problem **3-Satisfiability**, abbreviated as 3-SAT ([9], see also [15]).

**Theorem 1.** **Non-Well-Indumatched Graphs** is NP-complete.

**Proof.** Obviously, the problem is in NP. To show that it is NP-hard, we construct a polynomial-time reduction from 3-SAT. Let $C = \{c_1, c_2, \ldots, c_m\}$ and $X = \{x_1, x_2, \ldots, x_n\}$ be an instance of 3-SAT. Without loss of generality, we may assume that no clause in $C$ contains a variable $x_i$ and its negation $\overline{x}_i$, since such a clause is satisfied by any truth assignment and therefore can be eliminated. Also, we may assume that $m = |C| \geq 2$.

We construct a graph $G = G_{(C,X)}$ with the vertex set $C \cup X'$, where $C' = \{c_1, c_2, \ldots, c_m\}$ and $X' = \{x_i, a_i, b_i, \overline{x}_i : i = 1, 2, \ldots, n\}$ are disjoint sets, in the following way:

- The set $C'$ induces a clique.
- The set $X'$ induces $n$ four-paths $P^i = (x_i, a_i, b_i, \overline{x}_i)$ (with the edges $x_ia_i$, $a_ib_i$ and $b_i\overline{x}_i$).
- For each clause $c_j = (l^1_j \lor l^2_j \lor l^3_j)$, introduce the three edges $c_1l^1_j$, $c_2l^2_j$, $c_3l^3_j$ between $C'$ and $X'$ in $G$.

The graph $G$ associated with the instance $(C,X)$ of 3-SAT, where $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $C = \{c_1 = (x_1 \lor x_2 \lor x_3), c_2 = (\overline{x}_2 \lor \overline{x}_3 \lor x_4), c_3 = (\overline{x}_1 \lor x_3 \lor x_5), c_4 = (\overline{x}_3 \lor \overline{x}_4 \lor x_5)\}$, is shown in Fig. 1.

It is clear that the graph $G = G_{(C,X)}$ can be constructed in time polynomial in $m = |C|$ and $n = |X|$. We denote $X_i = \{x_i, a_i, b_i, \overline{x}_i\}$, $i = 1, 2, \ldots, n$.

**Claim 1.** For an arbitrary maximal induced matching $M$ in $G$, the following statements hold:

(i) $|EN_G(C') \cap M| \leq 1$,
(ii) $EN_G(X_i) \cap M \neq \emptyset$ for each $i = 1, 2, \ldots, n$, and
(iii) $|M| \in \{n, n + 1\}$. 


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Fig. 1. An illustration of the construction.

Proof. (i) Since $C'$ induces a clique, each induced matching in $E_N G(C')$ has at most one edge, i.e., $|E_N G(C') \cap M| \leq 1$.

(ii) If $M$ and $E_N G(X_i)$ are disjoint for some $i$ ($1 \leq i \leq n$), then we can add the edge $a_ib_i$ to $M$, contradicting the maximality of $M$.

(iii) By (i), $|E_N G(C') \cap M| \leq 1$. Also, $M$ contains at most $n$ edges of $G(X')$. Thus, $|M| \leq n + 1$. It follows from (ii) that $|M| \geq n$. □

Note that $G$ has a maximum induced matching of size $n + 1$, for example $\{c_1c_2\} \cup \{a_ib_i : 1, 2, \ldots, n\}$. Recall that $m = |C| \geq 2$.

We show that $C$ is satisfiable if and only if the graph $G$ has a maximal induced matching of size $n$. First, suppose that there exists a truth assignment $\phi$ satisfying $C$. We construct an induced matching $M \subseteq \{x_ia_i, b_i\overline{x}_i : i = 1, 2, \ldots, n\}$ choosing the $n$ edges that correspond to true literals under $\phi$. That is, if $\phi(x_i) = 1$, we include the edge $x_ia_i$ into $M$, otherwise the edge $b_i\overline{x}_i$ is included into $M$. Since $\phi$ satisfies $C$, each vertex of $C'$ is adjacent to an end-vertex of an edge in $M$. It implies that we cannot extend $M$ by adding an edge $e \notin E(G(X'))$. Indeed, such an edge is always incident to a vertex of $C'$. Thus, $M$ is a maximal induced matching of size $n$.

Conversely, let $M$ be a maximal induced matching in $G$ of size $n$.

Claim 2. Relation $M \subseteq E(G(X'))$ holds.

Proof. Claim 1 (ii) and $|M| = n$ imply that $M$ contains exactly one edge from each set $E_N G(X_i)$. Moreover, $M$ cannot have an edge connecting a vertex of $X'$ with a vertex of $C'$. Indeed, suppose that an edge $x_ic_j$ is in $M$. By Claim 1 (i), $M \setminus \{x_ic_j\} \subseteq E(G(X'))$. The vertex $\overline{x}_i$ is non-adjacent to both $x_i$ and $c_j$, since the clause $c_j$ cannot contain both $x_i$ and $\overline{x}_i$. Also, $b_i$ is non-adjacent to both $x_i$ and $c_j$. It follows that $M \cup \{b_i\overline{x}_i\}$ is an induced matching, a contradiction to the maximality of $M$. □

We say that a vertex $c_j \in C'$ is dominated by $M$ if $c_j$ is adjacent to an end-vertex of an edge in $M$.

Claim 3. At most one vertex in $C'$ is not dominated by $M$. 

Proof. Suppose that \( c_1 \) and \( c_2 \) are not dominated by \( M \). Then \( M \cup \{c_1c_2\} \) is an induced matching, contradicting to the maximality of \( M \).

Claim 4. There exists a maximal induced matching \( M' \subseteq E(G(X')) \) that dominates all vertices in \( C' \).

Proof. If \( M \) dominates all vertices in \( C' \), then we may set \( M' = M \), since \( M \subseteq E(G(X')) \) according to Claim 2. Otherwise, by Claim 3, we may assume that \( c_1 \) is the only vertex in \( C' \) that is not dominated by \( M \). Without loss of generality, let \( c_1 = x_1 \lor x_2 \lor x_3 \). By the maximality of \( M \), we cannot add the edge \( b_i \overline{x}_i \) to \( M \) for each \( i = 1, 2, 3 \). Since \( M \subseteq E(G(X')) \) and \( x_ia_i \notin M \), the edges \( a_ib_i \) are in \( M \), \( i = 1, 2, 3 \). Now we define \( M' = (M \setminus \{a_1b_1\}) \cup \{a_1a_1\} \). Since both \( a_1 \) and \( b_1 \) do not dominate any vertices in \( C' \), the induced matching \( M' \) dominates all vertices in \( C' \). In particular, \( M' \) is maximal. \( \square \)

The matching \( M' \) of Claim 4 covers at most one of the vertices \( x_i, \overline{x}_i \) for each \( i = 1, 2, \ldots, n \). Hence, we can define a partial truth assignment \( \phi' \) satisfying \( C \) letting a literal be true if the corresponding vertex of \( C' \) is covered by \( M' \). It remains to extend \( \phi' \) to a full assignment. This completes the proof of Theorem 1. \( \square \)

A graph \( G \) is said to be bi-size indumatched if there exists an integer \( k \geq 1 \) such that \( |M| \in \{k, k + 1\} \) for every maximal induced matching \( M \) in \( G \). Theorem 1 implies the following interesting corollaries.

Corollary 1. Non-Well-Indumatched Graphs is an NP-complete problem even for bi-size indumatched graphs.

Corollary 2. Non-Well-Indumatched Graphs is NP-complete for \((2P_5, K_{1,5})\)-free graphs.

Corollary 3. The decision problem corresponding to the problem of computing \( \sigma(G) \) is NP-complete within bi-size indumatched graphs.

Let \( WIM(t) \) be the class of graphs having maximal induced matchings of at most \( t \) sizes. Note that, if \( t = 1 \), then \( WIM(1) \) is the class of well-indumatched graphs. By means of Theorem 1, one can obtain the following result.

Theorem 2. For any positive integer \( t \), the problem of recognizing the class \( WIM(t) \) is co-NP-complete even for \((2P_5, K_{1,5})\)-free graphs.

3. NP-completeness results for well-indumatched graphs

A set \( I \subseteq V(G) \) is called independent or stable if no two vertices in \( I \) are adjacent. The independence number of a graph \( G \), denoted \( \alpha(G) \), is the maximum cardinality of an independent set in \( G \). A set \( D \subseteq V(G) \) is a dominating set if each vertex in \( V(G) \setminus D \) is adjacent to a vertex of \( D \). The minimum cardinality of a dominating set in \( G \) is the domination number of \( G \), denoted by \( \gamma(G) \). A set \( I \subseteq V(G) \) is called an independent dominating set if \( I \) is an independent set and \( I \) is a dominating set. The minimum cardinality of an independent dominating set of \( G \) is the independent domination number, and it is denoted by \( i(G) \).
The following three decision problems are known to be NP-complete (see, e.g., [15]).

**Independent Set**

*Instance:* A graph $G$ and an integer $k$.
*Question:* Is $\alpha(G) \geq k$?

**Dominating Set**

*Instance:* A graph $G$ and an integer $k$.
*Question:* Is $\gamma(G) \leq k$?

**Independent Dominating Set**

*Instance:* A graph $G$ and an integer $k$.
*Question:* Is $i(G) \leq k$?

Here we prove that Independent Set and Independent Dominating Set are NP-complete problems for well-indumatched graphs. Below we show that the Dominating Set problem is NP-complete even in a hereditary subclass of well-indumatched graphs.

**Theorem 3.** Independent Dominating Set is an NP-complete problem for well-indumatched graphs.

*Proof.* Clearly, the problem belongs to NP. To show that it is NP-hard, we establish a polynomial-time reduction from 3-Sat. For any instance $(C,X)$ of 3-Sat with the clauses $C = \{c_1,c_2,\ldots,c_m\}$ and variables $X = \{x_1,x_2,\ldots,x_n\}$, we construct a graph $G$ as follows.

- For each variable $x_i$, we introduce a complete graph $F_i = K_4$ with two special vertices $x_i$ and $\overline{x_i}$ called the literal vertices.
- For each clause $c_j$, we introduce a graph $H_j = P_7 = (a_j,b_j,c_j,d_j,a'_j,b'_j,d'_j)$, where $c_j$ is called the clause vertex.
- Edges connecting $V(F_i)$ and $V(H_j)$ are defined as follows: a clause vertex $c_j$ is connected to the three literal vertices corresponding to the literals in the clause $c_j$.

It is easy to see that the graph $G$ can be constructed in time polynomial in $m = |C|$ and $n = |X|$.

**Claim 5.** Each maximal induced matching $M$ in $G$ has exactly $2m + n$ edges.

*Proof.* Clearly, $M$ contains exactly one edge from each set $EN_G(V(F_i))$, $i = 1,2,\ldots,n$. Also, $M$ contains exactly two edges from each path $H_j$, $j = 1,2,\ldots,m$. □

**Claim 6.** There exists a satisfying truth assignment for $C$ if and only if $G$ has an independent dominating set of size $k = 2m + n$.

*Proof.* First, suppose that $C$ has a satisfying truth assignment. We construct an independent dominating set $I$ in $G$ as follows. If $x_i$ is assigned value 1, then include the literal vertex $x_i$ into $I$; otherwise, $\overline{x_i}$ is included into $I$. Finally, the set $\{b_j,b'_j:j = 1,2,\ldots,m\}$ is included into $I$. It is straightforward to verify that $I$ is an independent dominating set in $G$ of cardinality $2m + n$, as required.

Conversely, suppose that $I$ is an independent dominating set with $|I| = 2m + n$. Clearly, $I$ contains exactly one vertex from each $F_i$. Also, $I$ must contain at least two vertices from each set $S_j = \{a_j,b_j,a'_j,b'_j\}$ to dominate $a_j$ and $a'_j$. Since $|I| = 2m + n$, $I$ has exactly two
vertices from each \( S_j \). To dominate \( d_j \) and \( d'_j \), we must have \( I \cap S_j = \{ b_j, b'_j \} \). However, the set \( S_j \) does not dominate \( c_j \), so the vertex \( c_j \) must be dominated by some variable vertex. Thus, we can define a truth assignment \( \phi : X \to \{0, 1\} \) by \( \phi(x_i) = 1 \) if \( x_i \in I \), and \( \phi(x_i) = 0 \) otherwise. \(\square\)

Claims 5 and 6 imply that INDEPENDENT DOMINATING SET is NP-complete for well-indumatched graphs. \(\square\)

We now prove that INDEPENDENT SET is NP-complete for well-indumatched graphs.

**Theorem 4.** INDEPENDENT SET is an NP-complete problem for well-indumatched graphs.

**Proof.** We give a polynomial-time reduction from the INDEPENDENT SET problem for arbitrary graphs. Given a graph \( G \) with the edge set \( E(G) = \{e_1, e_2, \ldots, e_m\} \), \( m \geq 1 \), construct a new graph \( G^* \) as follows. First, for each edge \( e_i = u_iv_i \), \( i = 1, 2, \ldots, m \), add a triangle \( (x_i, y_i, z_i) \), and join \( x_i \) to \( u_i \) and \( y_i \) to \( v_i \), respectively. Thus, \( V(G^*) = V(G) \cup \{x_i, y_i, z_i : i = 1, 2, \ldots, m\} \) and \( E(G^*) = \{e_i, x_iu_i, y_iv_i, x_iz_i, x_iz_i, x_iz_i : i = 1, 2, \ldots, m\} \).

**Claim 7.** \( G^* \) is a well-indumatched graph.

**Proof.** Let \( G^*_i = G^*(\{u_i, v_i, x_i, y_i, z_i\}) \), \( i = 1, 2, \ldots, m \). An arbitrary maximal induced matching \( M \) contains at most one edge of each \( G^*_i \), therefore \( |M| \leq m \). Suppose that \( M \) and \( E(G^*_i) \) are disjoint for some \( i \in \{1, 2, \ldots, m\} \). By symmetry, we may assume that \( M \) does not contain any edge that is incident to \( u_i \). It implies that \( M \cup \{x_iz_i\} \) is an induced matching, a contradiction to the maximality of \( M \). Thus, \( |M| = m \). \(\square\)

**Claim 8.** \( \alpha(G^*) = \alpha(G) + m \), where \( m = |E(G)| \).

**Proof.** Let \( Z = \{z_1, z_2, \ldots, z_m\} \). If \( I \) is a maximum independent set in \( G \), then \( I^* = I \cup Z \) is an independent set in \( G^* \). Hence, \( \alpha(G^*) \geq |I^*| = \alpha(G) + m \). Conversely, let \( I^* \) be a maximum independent set in \( G^* \). Without loss of generality, we may assume that \( Z \subseteq I^* \). The set \( I = I^* \setminus Z \) is independent in \( G \), therefore \( \alpha(G) \geq |I| = \alpha(G^*) - m \). \(\square\)

Claims 7 and 8 imply that the described mapping of an instance \( (G, k) \) to the instance \( (G^*, k + |E(G)|) \) gives the required reduction, and the result follows. \(\square\)

The transformation used in the proof of Theorem 4 yields the following result for well-indumatched graphs.

**Corollary 4.** PARTITION INTO TRIANGLES and CHORDAL GRAPH COMPLETION are NP-complete problems for well-indumatched graphs.

Let \( H \) be a given graph. Consider the following decision problem.

**PARTITION INTO SUBGRAPHS ISOMORPHIC TO \( H \)**

**Instance:** A graph \( G \) with \( |V(G)| = q|V(H)| \) for some integer \( q \).

**Question:** Is there a partition \( V_1 \cup V_2 \cup \ldots \cup V_q = V(G) \) such that \( G(V_i) \) contains a subgraph isomorphic to \( H \) for all \( i = 1, 2, \ldots, q \)?

It is well known that this problem is NP-complete for each \( H \) that contains a connected component of three or more vertices ([17, 18], see also [15]).
Theorem 5. Partitions into Subgraphs Isomorphic to $P_3$ is NP-complete for well-indumatched graphs.

Proof. We give a polynomial-time reduction from the problem with $H = P_3$ for arbitrary graphs. Given a graph $G$ with $|V(G)| = 3q$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$, $m \geq 1$, construct a new graph $G^{**}$ as follows. First, for each edge $e_i = u_i v_i$, $i = 1, 2, \ldots, m$, add a new edge $x_i y_i$ and join $x_i$ to $u_i$ and $y_i$ to $v_i$, respectively. Then attach two pendant vertices $z'$ and $z''$ to each $z \in \{x_i, y_i\}$. More precisely,

$$V(G^{**}) = V(G) \cup \{x_i, x'_i, x''_i, y_i, y'_i, y''_i : i = 1, 2, \ldots, m\}$$

and $E(G^{**})$ consists of

- $e_i$ for all $i = 1, 2, \ldots, m$,
- $x_i u_i, y_i v_i$, for all $e_i = u_i v_i$, $i = 1, 2, \ldots, m$, and
- $x_i y_i, x_i x'_i, x_i x''_i, y_i y'_i, y_i y''_i$ for all $i = 1, 2, \ldots, m$.

It is easy to see that each maximal induced matching contains exactly one edge from each subgraph induced by $\{u_i, v_i, x_i, x'_i, x''_i, y_i, y'_i, y''_i\}$. Thus, $G^{**}$ is a well-indumatched graph. Also, $G$ has a partition into subgraphs $P_3$ if and only if $G^{**}$ has a partition into subgraphs $P_3$. Indeed, all 3-paths $(x'_i, x_i, x''_i)$ and $(y'_i, y_i, y''_i)$ are always in a partition of $G^{**}$ into 3-paths. □

Kobler and Rotics [20] proved that computing $\Sigma$ is NP-hard for line graphs and therefore, for claw-free graphs. One interesting special case of the problem is when the input line graph $L(G)$ is obtained from a graph $G$ with polynomial-time computable $\Sigma$. Corollary 5 shows that this special case is also NP-complete even if restricted to Hamiltonian line graphs of well-indumatched graphs.

Corollary 5. Computing $\Sigma$ for Hamiltonian line graphs $L(G)$ is NP-hard even if $G$ is a well-indumatched graph.

Kobler and Rotics [20] also proved that computing $\Sigma$ is NP-hard for Hamiltonian graphs. This proof is not related to their proof for claw-free graphs. Corollary 6 implies a unified result.

Corollary 6. Computing $\Sigma$ is NP-hard for Hamiltonian claw-free graphs.

The reduction used in the proof of Theorem 5 also gives the following result.

Corollary 7. Graph $k$-Colorability and Clique are NP-complete problems for well-indumatched graphs.

4. Co-indumatching subgraphs

The class $\mathcal{WIM}$ of well-indumatched graphs is not hereditary. For example, the path $P_7$ is a well-indumatched graph, while $P_5$ is not. However, $P_7$ contains $P_5$ as an induced subgraph. We introduce a new hereditary system that is similar to that for well-covered graphs [7, 31].
The neighborhood of an edge \( e = uv \) is defined as \( N(e) = N(u) \cup N(v) \). Clearly, \( u, v \in N(e) \). Also, \( N(S) = \cup_{e \in S} N(e) \) for a set \( S \) of edges. Let \( M \) be an induced matching in \( G \), including \( M = \emptyset \). A subgraph \( G - N(M) \) of \( G \) is called co-indumatching. We denote by CIMSub(\( G \)) the set of co-indumatching subgraphs in \( G \). For example, the cycle \( C_8 \) has the following co-indumatching subgraphs: \( C_8, P_4, K_1, K_0 \) and thus, CIMSub(\( C_8 \)) = \{\( C_8, P_4, K_1, K_0 \)\}. A class of graphs \( M \) is co-indumatching hereditary if CIMSub(\( G \)) \( \subseteq M \) for each graph \( G \in M \). A minimal forbidden co-indumatching subgraph for a co-indumatching hereditary class \( M \) is a graph \( F \) such that CIMSub(\( F \)) \( \setminus M = \{F\} \).

**Proposition 1.** \( WIM \) is a co-indumatching hereditary class.

**Proposition 2.** Each co-indumatching hereditary class can be characterized in terms of forbidden co-indumatching subgraphs.

**Proposition 2** means that for each co-indumatching hereditary class \( M \), there is a set of graphs \( \mathcal{X} \) such that \( G \in M \) if and only if \( G \) does not contain each graph in \( \mathcal{X} \) as a co-indumatching subgraph. Such a set \( \mathcal{X} \) must contain all minimal forbidden co-indumatching subgraphs for \( M \). Conversely, every set \( \mathcal{X} \) that contains all minimal forbidden co-indumatching subgraphs for \( M \) characterizes \( M \).

For example, \( P_5 \) is a forbidden co-indumatching subgraph for \( WIM \). Indeed, it is not well-indumatched, but \( P_5 - N(e) \) is well-indumatched for each edge \( e \) of \( P_5 \). We shall characterize well-indumatched graphs in terms of forbidden co-indumatching subgraphs.

Another important problem concerns the complexity of recognizing co-indumatching subgraphs.

**Co-Indumatching Subgraph**

**Instance:** A graph \( G \) and a set \( U \subseteq V(G) \) that induces a subgraph \( H \).

**Question:** Is \( H \) a co-indumatching subgraph of \( G \)?

We prove that the problem is hard.

**Theorem 6.** Co-Indumatching Subgraph is an NP-complete problem.

Let \( G_1, G_2, \ldots, G_k \) be pairwise vertex-disjoint graphs. The join of graphs \( G_1, G_2, \ldots, G_k \) is the graph \( \sum_{i=1}^{k} G_i = \overline{G_1} \cup \overline{G_2} \cup \cdots \cup \overline{G_k} \), where \( \overline{G_i} \) is the complement of \( G_i \). We denote by \( Z_W \) the set of all graphs of the form \( \sum_{i=1}^{k} G_i \), where each graph \( G_i \) is well-covered and \( \alpha(G_i) \neq \alpha(G_j) \) for \( 1 \leq i \neq j \leq k \). It is easy to see that \( Z_W \) consists of non-well-covered graphs. Let \( Z_{WIM} \) be the class of all graphs \( F \) such that \( (L(F))^2 \in Z_W \).

Here is our characterization.

**Theorem 7.** \( Z_{WIM} \) is the set of all minimal forbidden co-indumatching subgraphs for the class \( WIM \).

5. Perfectly well-indumatched graphs

In this section, we consider a hereditary subclass of the class \( WIM \). A graph \( G \) is perfectly well-indumatched if every induced subgraph of \( G \) is well-indumatched. We characterize perfectly well-indumatched graphs in terms of forbidden induced subgraphs.
**Theorem 8.** For a graph $G$, the following statements are equivalent:

(i) $G$ is a perfectly well-indumatched graph.

(ii) $G$ is a $(P_5,kite,butterfly)$-free graph.

(iii) $(L(G))^2$ is a $(K_4-e)$-free graph.

Theorem 8 implies that the class of perfectly well-indumatched graphs contains all $2K_2$-free graphs and therefore, all split graphs. Recall that Foldes and Hammer [14] characterized split graphs as $(2K_2,C_4,C_5)$-free graphs. It is well known that DOMINATING SET is an NP-complete problem for split graphs [2].

**Corollary 8.** DOMINATING SET is an NP-complete problem for perfectly well-indumatched graphs.

As we have shown, INDEPENDENT SET and INDEPENDENT DOMINATING SET are NP-complete problems for well-indumatched graphs. However, they can be solved in polynomial time for perfectly well-indumatched graphs.

**Theorem 9.** The INDEPENDENT SET problem and the INDEPENDENT DOMINATING SET problem can be solved in polynomial time for perfectly well-indumatched graphs, even in their weighted versions.

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**References**


Ecole Polytechnique, Laboratoire d’Informatique (CNRS LIX), F-91128 Palaiseau, France
E-mail address: philippe.baptiste@polytechnique.fr

United Institute of Informatics Problems, National Academy of Sciences of Belarus, 6 Surganova Str., 220012 Minsk, Belarus
E-mail address: kovalyov_my@newman.bas-net.by

Department of Discrete Mathematics and Algorithmics, Faculty of Applied Mathematics and Computer Science, Belarusian State University, Nezavisimosti Ave. 4, 220030 Minsk, Belarus
E-mail address: orlovich@bsu.by

Institute of Mathematical Optimization, Otto-von-Guericke-University of Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany
E-mail address: frank.werner@mathematik.uni-magdeburg.de

Department of Mathematical Cybernetics, Faculty of Mechanics and Mathematics, Belarusian State University, Nezavisimosti Ave. 4, 220030 Minsk, Belarus
E-mail address: zverovich@bsu.by