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Übungsaufgaben zur Vorlesung "Numerik der Navier-Stokes Gleichungen"
Wintersemester 2017/18 - Blatt 1

1. We consider Couette flow and Poiseuille flow in a channel. Show that both satisfy the *do-nothing* outflow condition on a straight outflow boundary orthogonal to the main flow direction

$$\partial_n \mathbf{v} - p\vec{n} = 0$$

(just consider $\rho = 1$ and $\nu = 1$). Do these two solutions also satisfy the full *no-stress* condition?

$$(\nabla \mathbf{v} + \nabla \mathbf{v}^T)\vec{n} - p\vec{n} = 0$$

2. A car has the length $L = 5m$. We build a model of size $L_M = 60$cm.

   a) which velocity $\mathbf{v}_M$ has to be considered in a wind tunnel experiment if the the car drives at $\mathbf{v} = 30$km/h, which velocity is required for $\mathbf{v} = 120$km/h.

   b) what happens if the experient is done under water? Which model velocities $\mathbf{v}_M$ are required?

3. Let $\Omega = (0,1)^2$ and $\gamma > 0$ the constant of the inf-sup condition

$$\inf_{p\in L^2(\Omega)\setminus\mathbb{R}} \sup_{\varphi\in H_0^1(\Omega)\setminus\{0\}} \frac{(p, \nabla\cdot\varphi)}{\|p\|\,\|\nabla\varphi\|} \geq \gamma > 0$$

   a) Show that the inf-sup condition does not change on a square of size $\Omega = (0, L)^2$ for $L > 0$

   b) Show that the inf-sup condition will change on anisotropic domains $\Omega = (0,1) \times (0, L)$

Hint: Construct a map $T : (0,1)^2 \to (0,L)^2$ or $T : (0,1)^2 \to (0,1) \times (0,L)$ and transform the integrals and derivatives .

4. We formulate the Stokes equation for $\mathbf{v} \in \mathcal{V} := H_0^1(\Omega)^d$ and $p \in \mathcal{L} := L^2(\Omega) \setminus \mathbb{R}$ as

$$\mathcal{A}\mathbf{v} + \operatorname{grad} p = \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0, \tag{1}$$

where $\mathcal{A} := -\Delta : \mathcal{V} \to L^2(\Omega)^d$ is a symmetric positive definite self-adjoint operator with a compact inverse (see the lecture). By $-\operatorname{grad} : \mathcal{L} \to \mathcal{V}_0^\circ$ we denote the weak gradient which is an isomorphism with adjoint $\operatorname{div} = -\operatorname{grad}'$ that itself is an isomorphism in $\operatorname{div} : (\mathcal{V}_0^\circ)' \to \mathcal{L}$.

We apply $-\operatorname{div} \mathcal{A}^{-1}$ to (1) and get

$$-\operatorname{div} \mathcal{A}^{-1} \operatorname{grad} p = -\operatorname{div} \mathcal{A}^{-1}\mathbf{f} + \underbrace{\operatorname{div} \mathbf{v}}_{=0}. \tag{2}$$

We define the operator

$$\mathcal{S} : \mathcal{L} \to \mathcal{L} \tag{3}$$

which is called the *Schur complement*.

a) Discuss the appearing function spaces. Is the transformation in (2) reasonable? Is it function space setting for $\mathcal{S}$ in (3) reasonable? Count the number of derivatives acting on p in (2). What is the "order" of the differential operator "$-\operatorname{div} \mathcal{A}^{-1} \operatorname{grad}$"?

b) The Schur component defines a bilinear form

$$s(p, q) := \langle \mathcal{A}^{-1} \operatorname{grad} p, \operatorname{grad} q \rangle_{H^1(\Omega) \times H^{-1}(\Omega)}.$$

Show, that $s(\cdot, \cdot)$ is continuous

$$s(p, q) \leq c \|p\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)} \quad \forall p, q \in \mathcal{L}$$

and elliptic

$$s(p, p) \geq \gamma \|p\|^2_{L^2(\Omega)} \quad \forall p \in \mathcal{L}$$

Hint: what is $l_p = -\operatorname{grad} p$ and what is $\mathbf{v}_p = \mathcal{A}^{-1} l_p$?

c) Given the results from b). Argue that there exists a unique $p \in \mathcal{L}$ for every $\mathbf{f} \in H^{-1}(\Omega)$.

Hint: Show that $-\operatorname{div} \mathcal{A}^{-1}\mathbf{f}$ defines a valid right hand side for the Schur complement problem. Discuss the application of Riesz representation theorem to this pressure problem.

Discussion *of the exercises on Friday, November 24. Please discuss the exercises during the class on Thursday, November 23 on your own. Please prepare notes for the Friday session!*