A modified global Newton solver for viscous-plastic sea ice models

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We present and analyze a modified Newton solver, the so called operator-related damped Jacobian method, with a line search globalization for the solution of the strongly nonlinear momentum equation in a viscous-plastic (VP) sea ice model.

Due to large variations in the viscosities, the resulting nonlinear problem is very difficult to solve. The development of fast, robust and converging solvers is subject to present research. There are mainly three approaches for solving the nonlinear momentum equation of the VP model, a fixed-point method denoted as Picard solver, an inexact Newton method and a subcycling procedure based on an elastic-viscous-plastic model approximation. All methods tend to have problems on fine meshes by sharp structures in the solution. Convergence rates deteriorate such that either too many iterations are required to reach sufficient accuracy or convergence is not obtained at all.

To improve robustness globalization and acceleration approaches, which increase the area of fast convergence, are needed. We develop an implicit scheme with improved convergence properties by combining an inexact Newton method with a Picard solver. We derive the full Jacobian of the viscous-plastic sea ice momentum equation and show that the Jacobian is a positive definite matrix, guaranteeing global convergence of a properly damped Newton iteration. We compare our modified Newton solver with line search damping to an inexact Newton method with established globalization and acceleration techniques. We present a test case that shows improved robustness of our new approach, in particular on fine meshes.

1. Introduction

Sea ice dynamics in polar regions play an important role in climate models. The extent of sea ice and open water as well as the thickness of the ice layer have a large impact on the climate system [30]. Heat and moisture exchange between the atmosphere and the ocean are strongly influenced by the ice layer that acts as an insulator. Furthermore, snow covered sea ice reflects a significant amount of the solar radiation while open water absorbs most of it. A shrinking ice layer will reduce the reflectivity and thus accelerate the reduction, the so called ice-albedo feedback [18, 30, 40, 47].

It has been shown by Kwok et al. [24] that many sea ice simulations significantly differ from satellite observations. It remains an open question how much of this discrepancy must be attributed to a
modeling error and how much to the numerical approximation error [18]. In this contribution, we will derive a robust and accurate numerical method to simulate the sea ice dynamics with a widely used sea ice model.

The choice of rheology, which describes the material behavior of sea ice influences the simulated ice deformation. Currently most sea ice models base on Hibler’s viscous-plastic (VP) rheology [28]. The VP rheology involves a strongly nonlinear dependency of the viscosities on the shear rates [11]. The treatment of the rheology term is still under active research. In the literature an implicit treatment of the nonlinear rheology term is proposed to properly simulate the deformation on the ice cover [17, 25].

Ip et. al [16] found that a fully explicit time stepping scheme for the momentum equation with VP rheology would require a prohibitively small time step of less than a second on a grid with 100km resolution. Therefore they proposed to use an implicit time stepping scheme [49]. A commonly used variant to solve the VP rheology was introduced by Hunke and Dukowicz [15], the elastic-viscous-plastic (EVP) model. Hunke and Dukowicz added an artificial elastic term to the VP rheology, which allows an explicit discretization at relatively large time steps. The additional elastic term produces noticeable errors in the sea ice dynamics compared to the use of implicit schemes for the VP model [3, 27, 31]. Lemieux et al. [27] modified the EVP solver by adding an inertia term to the momentum equation to compare the EVP and the VP solutions. Bouillon et al. [3] formulated the modified EVP solver as an iterative process converging to the VP solution. Kimmritz et al. [20] revised the convergence analysis of the modified EVP and showed for a simplified model that the EVP solution converges to the VP solution, if a sufficiently large number of subcycling steps is taken. Convergence can be accelerated by an adaptive control of the subcycling [19].

Apart from the explicit EVP there are two implicit schemes discussed in the literature to solve implicit discretizations of the momentum equation with the VP rheology. First, a fixed-point method, denoted as Picard solver, second an inexact Newton method. The Picard iteration was presented by Zhang and Hibler [49]. The idea of the standard Picard solver is to repeatedly solve a simple linearized system of equations. Lemieux and Tremblay [18] showed that the convergence of the Picard solver is slow. Therefore they developed the second approach, an inexact Newton method, realized as the Jacobian-free Newton-Krylov (JFNK) solver [28, 26].

The EVP and the Picard approach suffer from their poor convergence behavior. Typically only a small number of Picard iterations or substeps of the EVP model are considered. Thus the approximated solution still has a large numerical error which is accumulated over time [26]. This may result in large discrepancies of the simulated sea ice dynamics [18]. Newton solvers on the other hand tend to have difficulties if the Jacobian is ill-conditioned, nearly singular or if good initial values are not at hand. Lemieux et. al [28] mentioned that

Nonlinear solvers such as the JFNK method tend to have difficulties when there are such sharp structures in the solution. This lack of robustness of both solvers is however a debatable problem as it mostly occurs for large required drops in the residual norm. [...] Globalization approaches for the JFNK solver, such as the line search method, have not yet proven to be successful. Further investigation is needed.

Convergence of sea ice solvers usually worsens if a high spatial resolution is required [18, 28, 27]. We develop a robust Newton solver that is able to cope with high resolution. To which extent a high resolution is needed will depend on the specific application of the simulation. Averaged quantities like the overall extent of the ice layer will make lower demands than detailed comparisons to fracture patterns achieved by satellite observations [17, 30].

In this manuscript we present a new modified Newton solver for the VP sea ice momentum equation, which accelerates convergence compared to a standard Newton solver. The operator-related damped
**Jacobian method** has only recently been applied to *yield stress fluids*, such as the viscous-plastic Bingham model, see Mandal, Ouazzi and Turek [33]. The modified Newton solver is based on an analytical evaluation of the Jacobian matrix. To our best knowledge this is the first time in the context of the VP sea ice equations that the analytical Jacobian matrix is presented. We proof that the full Jacobian of the momentum equation is positive definite guaranteeing the global convergence of a properly damped Newton iteration. Further, we show that the Jacobian $A$ can be split into a positive definite part $A_1$, which is assumed to give stable convergence and a negative semidefinite part $A_2$, which is possibly troublesome. The idea of our solver is to combine a Picard solver with a Newton method. We consider $A_δ := A_1 + δA_2$, where $δ ∈ [0, 1]$, as the Jacobian of the modified Newton solver. Choosing $δ$ close to zero yields the stability of the Picard iteration, $δ$ close to one yields rapid Newton convergence. By an adaptive control of the parameter $δ$ (depending on the convergence history of the implicit scheme), we reach automatic transition to fast quadratic Newton convergence. Globalization of the modified Newton solver is done by damping the Newton iteration with a line search approach. Applied to an idealized sea ice test case, this new Newton strategy shows better robustness than a Newton method with global relaxation. We are able to reduce the required number of iterations by up to 80%. We aim to derive an efficient and robust method for directly tackling the VP model.

Experience with similar models involving complex rheology like reactive flows [4], fluids of power law type [8, 12], poroelastic flows [36] or fluid structure interactions [42] always shows superiority of a direct application of Newton’s method over any kind of linearized iterations. This mainly stems from the potential quadratic convergence and the possibility to robustly use large time steps.

The outline of the paper is as follows. We start with a description of the governing equations and present the viscous-plastic sea ice model in strong and variational formulation. Section 3 presents the discretization in space and time. Furthermore, to describe the finite element discretization, we present in detail the derivation of the Gâteaux derivative of the variational formulation, which will serve as basis for the modified Jacobian in the Newton iteration. Section 5 deals with different globalization and acceleration strategies for the inexact Newton solver. We present a line search approach and a crude approximation strategy of the linear subproblems, which are used in the JFNK solver, as well as our modified Newton solver. In Section 6 we present a test case for the sea ice equation and evaluate it on different mesh levels. In Section 7 we compare the robustness of the globalization and acceleration strategies and analyze the new operator-related damped Jacobian method. We conclude in Section 8.

### 2. Problem formulation

In this work, we consider a simplification of the sea ice model introduced by Hibler [11]. Thermodynamical effects and couplings to ocean circulation and atmospheric models are not taken into account. Sea ice is modeled by means of prescribing the ice concentration $A ∈ [0, 1]$ and the average ice height $H ≥ 0$. The ice concentration $A$ describes the fraction of grid cells covered by ice. The velocity $v$ is governed by a momentum equation

$$
ρ_{\text{ice}} H \left( \partial_t v + f_e e_r \times v \right) - \text{div} \ σ - τ_{\text{ocean}}(v) - τ_{\text{atm}} + ρ_{\text{ice}} H g \nabla \tilde{H}_d = 0,
$$

with the ice density $ρ_{\text{ice}}$, the forcing by ocean current $τ_{\text{ocean}}$ and the atmospheric flow $τ_{\text{atm}}$. The Coriolis parameter is denoted by $f_e$, $e_r$ is the radial unit vector, $\tilde{H}_d$ the surface height and $g$ the gravity. Ice is assumed to be a compressible fluid, which follows a nonlinear viscous-plastic (VP) constitutive law [11]. The stresses $σ$ are related to the strain rate tensor $\dot{e}$ by

$$
σ = 2η(\dot{e}, P)\dot{e} + ζ(\dot{e}, P) \text{tr}(\dot{e}) I - \frac{P}{2} I, \quad \dot{e} = \frac{1}{2}(\nabla v + \nabla v^T), \quad \dot{e}' = \dot{e} - \frac{1}{2} \text{tr}(\dot{e}) I,
$$

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with the trace-free deviatoric strain rate $\dot{\epsilon}'$ and the identity matrix $I$. With constants $P^*$ and $C$ the maximum ice strength \( P = P(A, H) \) is given by
\[
P(A, H) = P^* H \exp (-C(1 - A)).
\] (3)

Following Tremblay and Mysak [47] the surface height can be approximated by the following equation:
\[
g \nabla \tilde{H} + f_c e_r \times v_{\text{ocean}} \approx 0.
\] (4)

Forcing by ocean current and atmospheric flow is modeled as
\[
\tau_{\text{atm}} = \rho_{\text{atm}} C_{\text{atm}} \| v_{\text{atm}} \| 2 v_{\text{atm}},
\]
\[
\tau_{\text{ocean}}(v) = \rho_{\text{ocean}} C_{\text{ocean}} \| v_{\text{ocean}} - v \| 2 (v_{\text{ocean}} - v),
\] (5)

where by $\| \cdot \|_2$ we denote the Euclidean norm [34]. $C_{\text{atm}}$ and $C_{\text{ocean}}$ are the air and water drag coefficients, $\rho_{\text{atm}}$ and $\rho_{\text{ocean}}$ the densities and $v_{\text{atm}}$ and $v_{\text{ocean}}$ the velocity fields of ocean and atmospheric flow. The model is further simplified by assuming zero turning angles [11].

Hibler derived the viscosities $\eta$ and $\zeta$ by applying a normal flow rule to an elliptical yield curve [11]:
\[
\eta(\dot{\epsilon}, P) = e^{-2} \zeta(\dot{\epsilon}, P), \quad \zeta(\dot{\epsilon}, P) = \frac{P}{2 \Delta(\dot{\epsilon})},
\] (6)

\[
\Delta(\dot{\epsilon}) = \sqrt{2e^{-2} \dot{\epsilon}' : \dot{\epsilon}' + \text{tr}(\dot{\epsilon})^2},
\]

where $e = 2$ is the eccentricity of the ellipsoid defining the material rheology. By
\[
A : B := \sum_{ij} A_{ij} B_{ij}
\]
we denote the inner product of two tensors. To avoid degeneration of the viscosities for $\Delta(\dot{\epsilon}) \to 0$, $\Delta(\dot{\epsilon})$ is limited by
\[
\Delta(\dot{\epsilon}) \geq \Delta_{\text{min}} := 2 \cdot 10^{-9} \text{s}^{-1}.
\] (7)

Limitation of $\Delta(\dot{\epsilon})$ relates to maximal values for the viscosities $\eta$ and $\zeta$ given by (6). In the regime of very small strain rates, sea ice behaves like a linear viscous material. For a smooth and differentiable transition between plastic and viscous behavior, we replace $\Delta(\dot{\epsilon})$ by (see also Hader and Kryscher [22])
\[
\Delta(\dot{\epsilon}) := \sqrt{(2e^{-2} \dot{\epsilon}' : \dot{\epsilon}' + \text{tr}(\dot{\epsilon})^2) + \Delta_{\text{min}}^2}.
\] (8)

However, we do not use the replacement pressure [32]. Ice concentration $A$ and height $H$ are governed by balance laws
\[
\partial_t A + \text{div} (v A) = S_A(A, H) \quad A(x, y, t) \in [0, 1],
\]
\[
\partial_t H + \text{div} (v H) = S_H(A, H) \quad H(x, y, t) \in [0, \infty).
\] (9)

The reaction terms $S_A$ and $S_H$ describe the thermodynamical effects of ice freezing and melting [45, 11]. Here we neglect thermodynamical effects such that $S_A = S_H = 0$.

### 2.1. Variational formulation

To use a finite element discretization, we derive a variational formulation of the set of equations. Let $\Omega \subset \mathbb{R}^2$ be the spatial domain and $I = [0, T] \subset \mathbb{R}$ the time interval of interest. We prescribe homogeneous Dirichlet conditions $v = 0$ on the whole boundary $\Gamma := \partial \Omega$. Then, velocity is found in
\[
v(t) \in V,
\]
where $V \subset H^1_0(\Omega)^2$ is a function space with adequate regularity of the VP operator. By $H^1_0(\Omega)$ we denote the usual Sobolev space of $L^2(\Omega)$-functions with weak first derivatives in $L^2(\Omega)$ and with trace zero on the boundary $\Gamma$. Up to now, there are no existence and regularity results on the VP sea ice model. The model has similarities to the regularized $p$-Laplacian

$$(\mu(\Delta^2_{min} + |\nabla u|^2)^{p-2} \nabla u, \nabla \phi) = (f, \phi).$$

The limit case $p \to 1$ is however not covered by theory [48]. Another similarity is found with the time dependent minimal surface problem, see [29] and chapters 6 and 7 of [1]. Here, for the scalar equation

$$(\partial_t u, \phi) + ((1 + |\nabla u|^2)^{-\frac{1}{2}} \nabla u, \nabla \phi) = (f, \phi)$$

existence of solutions in the space $u \in W^{1,1}(\Omega)$ is shown. We believe that this result can be extended to the VP sea ice model as long as $A, H$ are smooth with $A, H > 0$, such that the operator does not degenerate. The major difficulty is to extend this scalar result to a system of equations.

Ice height and the ice concentration are found in

$$H(t) \in V^H := \{ \phi \in L^2(\Omega), \ v(t) \cdot \nabla \phi \in L^2(\Omega), \ \phi \geq 0 \ \text{a.e.} \},$$

$$A(t) \in V^A := \{ \phi \in L^2(\Omega), \ v(t) \cdot \nabla \phi \in L^2(\Omega), \ 0 \leq \phi \leq 1 \ \text{a.e.} \}.$$ We multiply the momentum equation (1) with $\phi \in V$ and apply partial integration. Further, the balance laws are multiplied with $\phi_H, \phi_A \in \mathcal{L} := L^2(\Omega)$ and integrated over $\Omega$, see [7]. Finally, for all $t \in [0, T]$ we find $v, H$ and $A$

$$(\rho_{\text{ice}} H \partial_t v, \phi)_\Omega + (f, e_r \times (v - v_{\text{ocean}}), \phi) - (\mathbf{\tau}_{\text{atm}}, \phi)_\Omega$$

$$- (\mathbf{\tau}_{\text{ocean}}(v), \phi)_\Omega + (\mathbf{\sigma}(v, H, A), \nabla \phi)_\Omega = 0 \ \forall \phi \in V,$$

$$\partial_t H + v \cdot \nabla H + H \text{div}(v), \phi_H)_\Omega = 0 \ \forall \phi_H \in \mathcal{L},$$

$$(\partial_t A + v \cdot \nabla A + A \text{div}(v), \phi_A)_\Omega = 0 \ \forall \phi_A \in \mathcal{L},$$

where by $(\cdot, \cdot)$ we denote the $L^2$ scalar product on $\Omega$. By $\| \cdot \|$ we denote the $L^2$ norm. The constraints $H(x, y, t) \geq 0$ and $A(x, y, t) \in [0, 1]$ are embedded in the trial-spaces $V^A$ and $V^H$ and must be realized by a projection of the solution.

3. Discretization and solution

We use finite elements for the spatial discretization, which is based on a quadrilateral mesh. The model is implemented in the software library Gascoigne [2]. A finite element scheme on triangular meshes is described by Timmermann et. al [46] and Danilov et. al [9]. The choice of quadrilateral hierarchical meshes mostly stems from the possibility to use efficient geometric multigrid solvers [35] for all linear problems.

3.1. Discretization of the balance laws and the momentum equation

We decouple the momentum equation and the balance laws. The decoupling is standard to cope with the complex system [9, 11, 14, 26]. For temporal discretization we start by introducing a discretization of the time span of interest $I = [0, T]$ into discrete uniform steps $I_n = [t_{n-1}, t_n],$

$$0 = t_0 < t_1 < \cdots < t_N = T, \ \ k := t_n - t_{n-1},$$

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where by $k$ we denote the step size. At time $t_n$ we denote by $v^n, A^n, H^n$ approximations to ice velocity, concentration and height. At time $t_0 = 0$, $v^0, A^0$ and $H^0$ are given as initial values.

In every time step $t_{n-1} \to t_n$, we first solve the balance laws (9). Accurate discretization of these linear transport equations is a cumbersome task in the context of continuous finite elements, as sharp layers and even discontinuities may appear in the solution that are difficult or even impossible to resolve. Here, the balance laws are treated with a flux limited Taylor-Galerkin scheme that adds spatial stability to the linear transport problems without artificial stabilization parameters [23]. As this method is explicit, we apply it in a subcycling approach by splitting the time-step $t_{n-1} \to t_n$ into a sequence of substeps. For details we refer to Danilov et. al [9], where a similar approach is described for a finite element sea ice model on triangular meshes (see also [35]).

The approximation of the balance laws is succeeded by solving the sea ice momentum equation (1). Our experience with the sea ice equations shows that temporal accuracy is of lesser importance compared to stability problems. Hence, we use the simple implicit Euler time stepping method (similar to Lemieux and Tremblay [28]). Therefore the momentum equation results in a nonlinear system of quasi-stationary differential equations. The nonlinearity coming from the viscous-plastic material tensor $\sigma (v^n)$ is severe and the efficient treatment of this term is still under active research [19]. We use a Newton linearization to achieve fast convergence. For the following, we write each time step using the abstract notation

$$A(v^n, \phi) = F(\phi),$$

where

$$A(v^n, \phi) := (\rho_{\text{ice}} H^n v^n, \phi) + k(f_e e_r \times v^n, \phi) + k(\sigma^n, \nabla \phi) - k(t_{\text{ocean}}(t_n, v^n), \phi),$$

$$F(\phi) := (\rho_{\text{ice}} H^n v^{n-1}, \phi) + k(t_{\text{atm}}(t_n), \phi) + k(f_e e_r \times v_{\text{ocean}}, \phi).$$

### 3.2. Newton scheme for the momentum equations

The general Newton method for the solution of variational problems like (11) is given as

$$w^{n,(l)} \in V, \quad A'(v^{n,(l-1)})(w^{n,(l)}, \phi) = F(\phi) - A(v^{n,(l-1)}, \phi), \quad \forall \phi \in V,$$

$$v^{n,(l)} = v^{n,(l-1)} + w^{n,(l)}, \quad l = 1, 2, \ldots$$

where by $v^{n,(0)}$ we denote an initial guess which is usually taken as $v^{n,(0)} = v^{n-1}$, the solution of the last time step. By $A'(v)(w, \phi)$ we denote the Gâteaux derivative of $A(v, \phi)$ in direction $w$, which is defined as

$$A'(v)(w, \phi) := \frac{d}{ds} A(v + sw, \phi) \bigg|_{s=0}. $$

The Gâteaux derivative is the generalization of the directional derivative in $\mathbb{R}^n$ to infinite dimensional function spaces like the Sobolev space $H^1_0(\Omega)$. Considering the discrete setting with a finite dimensional space $V_h$, the notations Gâteaux derivative and directional derivative coincide. This derivative can be approximated by finite differences [21], expressed by automatic differentiation [39] or derived analytically. Here an analytical evaluation of the Jacobian is used for a better understanding of its structure.

Although the VP model involves strong nonlinearities and a complex viscosity structure, the resulting Jacobian is positive definite matrix and well-structured. For functions with a positive definite and Lipschitz continuous Jacobian, a damped Newton method is globally convergent (see Lemma 3.5 of [10]). We use a line search method to damp each Newton iteration, see Section 4.2. The following theorem describes the different properties of the Jacobian.
Theorem 4 (Jacobian of the VP model) The derivative $A'(v)(w, \phi)$ of the implicit Euler discretization of the VP model (12) is given by

$$A'(v)(w, \phi) = (\rho_{ice} H w, \phi) + k(f_e e_r \times w, \phi)$$

$$+ k(\sigma'_1(w), \nabla \phi) + k(\sigma'_2(w), \nabla \phi) - k(\tau'_{ocean}(w), \phi),$$

where the derivatives of the stress-tensor $\sigma' = \sigma'_1 + \sigma'_2$ with respect to strain rate $\sigma'_1$ and with respect to the viscosities $\sigma'_2$ and the derivatives of the ocean forcing $\tau'_{ocean}$ are given by

$$\sigma'_1(w) = 2e^{-2}\zeta\epsilon'(w) + \zeta \text{tr}(\epsilon'(w)) I,$$

$$\sigma'_2(w) = -\frac{\zeta}{\Delta^2} \left(2e^{-2}\epsilon'(w) + \text{tr}(\epsilon'(w)) \right) \left(2e^{-2}\epsilon' + \text{tr}(\epsilon(I))\right),$$

$$\tau'_{ocean}(w) = -\rho_{ocean} C_{ocean} \left(\|v_{ocean} - v\|_\Omega + \frac{(v_{ocean} - v) \cdot w}{\|v_{ocean} - v\|_\Omega} - (v_{ocean} - v)\right).$$

Apart from the Coriolis term the Jacobian is symmetric

$$A'(v)(w, \phi) = A'(v)(\phi, w) - 2k(f_e e_r \times \phi, w).$$

Furthermore, $\sigma'_1$, as well as the complete Jacobian are positive definite, while $-\tau'$ is positive semidefinite and $\sigma'_2$ is negative semidefinite. It holds

$$c \min_\Omega \left\{ \Delta^2_{\min} \right\} \|\sqrt{\zeta} w\|^2_{H^1(\Omega)} \leq A'(v)(w, w) \leq c \|\sqrt{\zeta} w\|^2_{H^1(\Omega)},$$

where the constant $c > 0$ depends on the domain $\Omega$, the time step size $k$, the material parameters and the velocity $v$. Further, $\|w\|^2_{H^1(\Omega)} := \|w\|^2 + \|\nabla w\|^2$.

**Proof:** All results can be shown by basic calculus. For better readability we give the proof in the appendix.

The theorem has several implications for the design of numerical methods for approximating the VP sea ice model. Inequality (16) shows that all eigenvalues of the Jacobian are bound from below and above. For $\zeta > 0$ (which is the case for $H > 0$), all eigenvalues are positive such that the matrix is positive definite. As mentioned above, this results in the global convergence (for all initial values) of a properly damped Newton scheme. We do not only include the case $\zeta = 2$ but also models, where $0 < \zeta < 1$ is chosen.

The ratio of the largest and the smallest eigenvalue indicates the condition number of the matrix. This number is important, as it describes the propagation of errors when solving linear systems. Usually, the convergence rate of iterative solvers worsens if the condition number gets large. The convergence rate of the Newton method depends on the smallest eigenvalue. This shows the role of limiting $\Delta \geq \Delta_{\min}$ in the lower bound of (16).

Finally, we showed that the Jacobian is symmetric if we neglect the Coriolis term. An explicit discretization of this term would allow to use the Conjugate Gradient method as linear solver. It is more efficient than a Generalized minimal residual method (GMRES). In particular, it does not require additional storage for orthogonalizations, see [44].

**Approximation of the linear subproblem** It remains to describe the solution of a linear system of equations $Ax = b$, where by $x$ we denote the coefficient vector of the two velocity components, by $A$ the (possibly damped) Jacobian and by $b$ the Newton residual, see (13). If the linear subproblem is not solved exactly, the Newton method is called an inexact Newton method. A common approach for the solution of an ill-conditioned, nondefinite system of equations is to employ a GMRES method, which will be preconditioned with a geometric multigrid solver [25, 44]. A detailed description of the linear solver can be found in [35].

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5. Globalization and acceleration methods

Existing solvers for the sea ice momentum equation are not robust with increasing resolution [28]. There are mainly two stabilization strategies to improve the numerical convergence properties of the JFNK solver. First, a crude and very imprecise approximation of the linear residuals and second, a line search approach [26]. Lemieux et al [32] suggested to test a combination of the JFNK with a Picard solver to further improve the convergence properties of the solver. A combination of a Newton method and Picard solver for multidiimensional variably saturated flow problems was done in [6]. First a Picard solver is used until the iteration converges steadily, then the Newton method is applied. While this approach has similarities to our proposed operator-related damped Newton scheme (Section 4.3), the adaptive control is different. A comparison of globalization and acceleration strategies presented in this Section is given in Section 7.

5.1. Crude approximation strategy of the linear subproblem

The crude approximation of the linear subproblem (17) is a stabilization technique that has been applied in the first version of the JFNK solver [28]. The JFNK solver is based on an inexact Newton method. In the $l$-th iteration of the Newton method (13) one requires

$$
\|A'(v^{n,(l-1)})(w^{n,(l)}) - R(v^{n,(l-1)})\| \leq \text{tol}(l)\|R(v^{n,(l-1)})\|,
$$

where the residual vector $R(v)$ of equation (11) is given by

$$
[R(v)]_i := F(\phi_i) - A(v, \phi_i), \quad [A'(v)(w)]_i := A'(v)(w, \phi_i),
$$

with the basis function $\phi_i \in V_h$. The crude approximation strategy of the linear subproblem is based on an adaptive control of the tolerances $\text{tol}(l)$. Lemieux and Tremblay [28] found that the convergence of the nonlinear solver is slow in the beginning and fast in the end of a time step. Additionally they observed that in some cases a high initial tolerance criterion can lead to a not converging Newton solver due to over-solving. Over-solving describes the effect that solving the linear subproblem up to small residual can cause an inaccurate nonlinear correction of the solution [41]. Therefore Lemieux and Tremblay suggested to choose the initial tolerance high and reduce it as one gets in the area of fast convergence [26]. The tolerances for the linear problems are chosen (similar to [43]) as

$$
\text{tol}(l) = \begin{cases} 
0.99, & \text{if } ||R(v^{n,(l-1)})|| \geq \frac{2}{3}||R(v^{n,(0)})||, \\
\left(\frac{||R(v^{n,(l-1)})||}{||R(v^{n,(l-2)})||}\right)^{1.5} & \text{else.}
\end{cases}
$$

Further, if the Newton iteration count exceeds $l > 100$, the tolerance is taken as $\text{tol}(l) = 0.99$ to improve robustness.

5.2. Line search method

To stabilize the JFNK solver a line search method is applied. The general Newton method with the line search approach for the solution of variational problems is given as

$$
\begin{align*}
\mathbf{w}^{n,(l)} & \in V \\
A'(v^{n,(l-1)})(\mathbf{w}^{n,(l)}, \phi) &= F(\phi) - A(v^{n,(l-1)}, \phi) \quad \forall \phi \in V, \\
v^{n,(l)} &= v^{n,(l-1)} + \omega \mathbf{w}^{n,(l)},
\end{align*}
$$

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By $\omega \in (0, 1]$ we denote the line search parameter to relax the Newton update $w^{n,(l)}$. This parameter is chosen as large as possible (close to one) but small enough, such that the Newton residual decays:

$$\omega \in \{1, (1-\gamma), (1-\gamma)^2, \ldots \} : \quad \|R(v^{n,(l-1)} + \omega w^{n,(l)})\| < \|R(v^{n,(l-1)})\|,$$

where $\gamma \in (0, 1)$. Lemieux et al [26] choose $\gamma = 0.5$ and do a maximum of 4 iterations of the line search approach. The line search method strongly increases the robustness of the JFNK [26].

As shown in Theorem 4, the Jacobian of the sea ice momentum equation is positive definite. For functions with a positive definite and Lipschitz continuous Jacobian a damped Newton method is globally convergent (see Lemma 3.5 of [10]). Therefore we apply a line search method as globalization strategy to the modified Newton solver. We choose $\gamma = 0.75$ for better robustness and perform a small number of line search iterations (maximum of 4), even if the residual will still be increased.

### 5.3. Operator-related damped Newton

To derive the modified Newton solver we follow [33]. The operator-related damped Jacobian method was introduced by Horn, Ouazzi and Turek [13, 38]. More details can be found in [37]. As presented in Section 3.2, the Jacobian $A'$ can be split into a positive definite part $A'_1$ and a negative semidefinite part $A'_2$,

$$A'(v)(w, \phi) = A'_1(v)(w, \phi) + A'_2(v)(w, \phi).$$

We refer to the positive definite part as the good, stabilizing part of the Jacobian

$$A'_1(v)(w, \phi) = (\rho_{ice} H w, \phi) + k(f, e_r \times w, \phi) + k(\sigma'_1(w), \nabla \phi) - k(\tau'_{ocean}(w), \phi),$$

whereas the negative semidefinite part is possibly troublesome

$$A'_2(v)(w, \phi) = k(\sigma'_2(w), \nabla \phi).$$

To improve robustness of the modified Newton method we adaptively control this negative semidefinite part of the Jacobian.

The reason for splitting the Jacobian is twofold. First, for differentiable functions with a positive definite, Lipschitz continuous Jacobian, a damped Newton method is globally convergent. The convergence rate of the damped Newton method with positive definite Jacobian depends on the bound of the inverse of the Jacobian, which is given by the inverse of its lowest eigenvalue. By decreasing the influence of $A'_2(v)(w, \phi)$ the lowest eigenvalue is increased and the convergence of the method is improved, see Theorem 1. A complete analysis of the operator-related damped Newton method is still subject to future work. A second benefit of adaptive controlling $A'_2(v)$ is the enhanced condition number of the system matrix (Section 3.2).

The Jacobian of the modified Newton solver is given as

$$A'_3(v)(w, \phi) := A'_1(v)(w, \phi) + \delta A'_2(v)(w, \phi),$$

where the parameter $\delta \in [0, 1]$ is adaptively chosen depending on the convergence history. The idea is to choose $\delta$ as 1 if the reduction rate of the Newton method is good. If not, or only very slow convergence is observed, we reduce $\delta$ to get closer to a robust fixed-point iteration, which for $\delta = 0$ is very similar to typical Picard solvers [28].

The choice of $\delta$ is crucial for the convergence rate of the modified Newton solver. With the residual (18) of the Newton scheme, we define the reduction rate in the $l$-th iteration of the Newton method as

$$Q_l := \frac{\|R(v^{n,(l)})\|}{\|R(v^{n,(l-1)})\|}.$$
Similar to Turek et al. [33] we choose the damping parameter in the $l$-th step as

$$
\delta(l) = \begin{cases} 
\min\{1, \delta(l-1) \left( 0.2 + \frac{4}{0.7 + \exp(1.5Q_l)} \right) \} & \delta(l-1) \geq 0.2, \\
1 & \delta(l-1) < 0.2.
\end{cases}
$$

(22)

This relation is found by experimental tuning. Other feedback functions give similar results without a distinct superiority. We therefore adapted the original suggestion. A constant choice of $\delta$ does not increase the convergence rate of the modified Newton solver. In opposition to Turek we start with $\delta(0) = 1$ to achieve rapid Newton convergence if possible. The damping parameter is reduced if the reduction rate exceeds $Q_l > \tilde{Q} \approx 0.97$, see Figure 1. Otherwise, we increase $\delta(l)$ up to $\delta(l) = 1$, which corresponds to the standard Newton scheme. To avoid very rapid changes of $\delta(l)$ we modify the parameter relative to the last value. In contrast to the original work of Turek, we reset $\delta(l)$ to 1, once $\delta(l-1)$ reaches a limit of 0.2, and restart the Newton iteration with probably better initial values. This is to avoid stagnation of convergence. A numerical analysis of the operator-related damped Newton method is given in Section 7.

6. Test case

Our test case is designed to have large deformations at the beginning of the simulation. A time-depending wind forcing will avoid stationary solutions. We present a simple analytical wind and ocean forcing that makes the test case easy to reproduce.

We consider the quadratic domain $\Omega = (0, 500 \text{ km})^2$ and simulate the sea ice dynamics for $T = 8 \text{ days}$. The time $t$ is measured in days. Similar to Hunke [14], we prescribe a circular steady ocean current

$$
v_{\text{ocean}} = v_{\text{ocean}}^{\text{max}} \left( \frac{2y/500 \text{ km} - 1}{1 - 2x/500 \text{ km}} \right),
$$

(23)

where the maximal ocean velocity is given as

$$
v_{\text{ocean}}^{\text{max}} = 0.01 \text{ ms}^{-1}.$$

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Cyclone at \( t = 0 \) days  
Anticyclone at \( t = 8 \) days  
Water stress

Figure 2: Forcing by wind and water. The center of the wind field moves in time. The water forcing is stationary.

The wind field is a cyclone followed by an anticyclone, diagonally passing from the mid point to the edge or the edge to the mid point of the computational domain:

\[
v_{\text{atm}}(t) = \bar{v}_{\text{atm}}^{\text{max}} \left( \begin{array}{cc} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{array} \right) \left( \begin{array}{c} x - m_x(t) \\ y - m_y(t) \end{array} \right) \omega(x, y, t).
\]

The wind forcing alternates from cyclonic to anticyclonic with a period of 4 days. The maximum wind velocity (in \( \text{m s}^{-1} \)) is given by

\[
\bar{v}_{\text{atm}}^{\text{max}}(t) = 15 \text{ ms}^{-1} \begin{cases} 
-\tanh\left(\frac{(4-t)(4+t)}{2}\right) & t \in [0, 4], \\
\tanh\left(\frac{(12-t)(t-4)}{2}\right) & t \in [4, 8].
\end{cases}
\]

The center of the cyclone is a function of time and given by

\[
m_x(t) = m_y(t) = \begin{cases} 
250 \text{ km} + 50t \text{ km/day} & t \in [0, 4], \\
650 \text{ km} - 50t \text{ km/day} & t \in [4, 8].
\end{cases}
\]

Divergence and convergence angles of anticyclone and cyclone are set to 9° and 18°, which corresponds to

\[
\alpha = \begin{cases} 
90^\circ - 18^\circ = 72^\circ & t \in [0, 4], \\
90^\circ - 9^\circ = 81^\circ & t \in [4, 8].
\end{cases}
\]

Finally, the factor \( \omega(x, y) \) is chosen to reduce the wind strength away from the center:

\[
\omega(x, y) = \frac{1}{50} \exp\left(-\frac{r(x, y)}{100 \text{ km}}\right), \quad r(x, y) = \sqrt{(x - m_x(t))^2 + (y - m_y(t))^2}.
\]

Snapshots of the wind and the water stresses are shown in Figure 2. As initial condition we use zero velocity, constant ice concentration \( A = 1 \) and a small variation of the ice height \( H = 0.3 \) m:

\[
v^0 = 0 \text{ m/s}, \quad A = 1, \quad H(x, y) = 0.3 \text{ m} + 0.005 \text{ m} \left( \sin\left(\frac{x}{2 \text{ km}}\right) + \sin\left(\frac{y}{2 \text{ km}}\right) \right).
\]

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<th>Value</th>
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</tr>
<tr>
<td>$\rho_{\text{atm}}$</td>
<td>air density</td>
<td>1.3 kg/m$^3$</td>
</tr>
<tr>
<td>$\rho_{\text{ocean}}$</td>
<td>water density</td>
<td>1026 kg/m$^3$</td>
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<tr>
<td>$C_{\text{atm}}$</td>
<td>air drag coefficient</td>
<td>$1.2 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$C_{\text{ocean}}$</td>
<td>water drag coefficient</td>
<td>$5.5 \cdot 10^{-3}$</td>
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<tr>
<td>$f_c$</td>
<td>Coriolis parameter</td>
<td>$1.46 \cdot 10^{-4}$ s$^{-1}$</td>
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<tr>
<td>$P^*$</td>
<td>ice strength parameter</td>
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<tr>
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<td>ice concentration parameter</td>
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</tr>
<tr>
<td>$\epsilon$</td>
<td>ellipse ratio</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: Physical parameters of the momentum equation.

At the boundary of the domain we use a no-slip condition for the velocity

$$v = 0$$ on $\partial \Omega$.

All further problem parameters are gathered in Table 1.

**Evaluation**  We analyze the test case on different mesh levels of width

$$\frac{500 \text{km}}{2^6} \approx 7.81 \text{ km}, \quad \frac{500 \text{km}}{2^7} \approx 3.91 \text{ km}, \quad \frac{500 \text{km}}{2^8} \approx 1.95 \text{ km}.$$  

For simplicity, we refer to these meshes as $8 \text{ km}$, $4 \text{ km}$ and $2 \text{ km}$. We use a time step of $k = 2000 \text{ sec} \approx 0.5 \text{ h}$. In Figure 3, we show the ice velocity field at times $t = 1, 2, 4, 6, 7, 8$ days. Figure 4 shows the bulk viscosity $\eta$ in logarithmic scale to the basis 10 under mesh refinement. The values of $\eta$ vary over nearly 10 magnitudes. As the mesh is refined the values of $\eta$ change more rapidly. This effect is most visible on the $2 \text{ km}$ mesh. The increasing variation of viscosities on high resolution meshes causes difficulties for the nonlinear solvers and makes the momentum equation very hard to solve. In applications, where these small scale patterns are of interest, we need a robust solver on fine meshes.

For dominant small scale patterns, the lower bound in (16) will decrease, as $\Delta(\dot{\epsilon})$ increases.

To evaluate the discretization of the test case we show, following Lemieux and Tremblay [18], the states of stress in the stress-invariant space $(\sigma_I/P, \sigma_{II}/P)$, with

$$\sigma_I = \zeta \dot{\epsilon}_I - \frac{P}{2}, \quad \sigma_{II} = \eta \dot{\epsilon}_{II},$$

$$\dot{\epsilon}_I = v_x^1 + v_y^2, \quad \dot{\epsilon}_{II} = \sqrt{(v_x^1 - v_y^2)^2 + (v_y^1 + v_x^2)^2}.$$  

Following Hibler [11] the viscous-plastic material behavior is described by an ellipse

$$E(\sigma_I/P, \sigma_{II}/P) := (2\sigma_I/P + 1)^2 + (4\sigma_{II}/P)^2 - 1 = 0.$$  

In Figure 5, we show the stress states in the stress-invariant space for different mesh levels. The maximal residual of equation (24) is bounded by $5 \cdot 10^{-6}$. While the maximal residual does not increase on fine meshes, the number of nodes with high residual is slightly enlarged. On a $2 \text{ km}$ mesh 1.7%, on a $4 \text{ km}$ mesh 1.4% and on a $8 \text{ km}$ mesh 0.8% of stress states lying outside of the elliptical
Figure 3: Ice velocity vector field (in \( \text{m s}^{-1} \)) and ice velocity streamlines. Evaluating the test case at \( t = 1, 2, 4, 6, 7, 8 \) days using a step size of \( k = 0.5 \text{ h} \) and a 2 km mesh.

Following Lemieux et. al [25] an approximation is considered to be a VP solution if the following conditions are fulfilled.

\[
1 \leq \frac{\sigma_I}{P} \leq 0 \quad \text{for at least 99\% of the } \sigma_I,
\]

\[
0 \leq \frac{\sigma_{II}}{P} \leq f\left(\frac{\sigma_I}{P}\right) + 0.005, \quad \text{with } f\left(\frac{\sigma_I}{P}\right) = 0.25 \sqrt{-\left(2\frac{\sigma_I}{P} + 1\right)^2 + 1}.
\]

We fulfill both criteria for every grid point on all mesh levels.

**Solution of the nonlinear and linear systems** In every time step, we solve the nonlinear momentum equation until either a global residual condition (as defined in (18)) is fulfilled

\[
\| \mathcal{R}(\mathbf{v}^{(l)}) \| \leq tol_g, \quad tol_g := 10^{-13},
\]

or until a relative reduction of the residual with respect to the initial residual is obtained

\[
\| \mathcal{R}(\mathbf{v}^{(l)}) \| \leq \| \mathcal{R}(\mathbf{v}^{(0)}) \| \cdot tol_r, \quad tol_r := 10^{-4}.
\]

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Figure 4: Plot of $\eta$ in a logarithmic scale to the basis 10. Evaluating the test case at $t = 2$ days using a time step size of $k = 0.5$ h. On a 2 km mesh the viscosities vary by nearly 10 orders of magnitude on very small spatial scales.

The linear problems $A[v^{(l-1)}]w = b$ arising within the Newton iteration (19) are approximated with the tolerance

$$\|b - A[v^{(l-1)}]w\| \leq 10^{-4}\|b\|.$$  

We limit the number of nonlinear steps to 200 iterations and the number of linear GMRES steps to 50 iterations.

7. Discussion

We compare the robustness of an inexact Newton method using the globalization and acceleration strategies of Section 5 - a line search approach, the crude approximation strategy of the linear subproblem and an operator-related damped Newton method. We solve the test problem of Section 6 and analyze the behavior of these three approximation strategies under mesh refinement. By refining the mesh solution structures get sharper and solving the implicit momentum equation is increasingly difficult. Furthermore we give an analysis of the new operator-related damped Newton method on different mesh levels. Linear and nonlinear tolerance are set as described in Section 6.

7.1. Comparison of the different globalization strategies

As shown in Figure 6 (top) on a 8 km mesh the inexact Newton method globalized by a line search approach with and without crude approximation strategy fails to reach the desired nonlinear residual after about 160 time steps. Instead, the operator-related damped Jacobian method with a line search globalization provides a robust strategy. On finer meshes this result becomes even clearer. As shown in Figure 6 (middle) on a 4 km mesh the inexact Newton method with a line search approach fails in two time steps. The inexact Newton method with a line search approach and the crude approximation strategy is more robust, but it is not converging after 6 time steps. Using the operator-related damped Jacobian approach with a line search globalization the inexact Newton method is robust for all time steps. On a 2 km, see Figure 6 (bottom), the inexact Newton method globalized by a line search approach with and without crude approximation strategy could not reduce the nonlinear residual to the required tolerance after two time steps. The operator-related damped Jacobian method with a
Figure 5: States of stress in the stress-invariant space. Evaluating the test case at $t = 2$ days using a time step size of $k = 0.5h$.

The line search strategy solves the test problem until time step 272 ($\approx 6$ days), where the desired tolerance could not be reached in the given 200 steps.

Here small scale patterns evolve, the term $\Delta^2_{min}/\Delta^2$ gets small and the lower bound of the Jacobian (16) turns towards zero. This could be one reason for the slower convergence of the modified Newton solver. If we however continue the simulation for the full temporal interval of $I = [0, 8 \text{ days}]$, we only note 6 failures out of 350 steps. In comparison, if a failure occurs using a line search with and without the crude approximation strategy the residual could not be reduced in the subsequent 10 time steps. Therefore the simulation was terminated.

It is typical to all existing implicit sea ice solvers [28] that the desired residual cannot be reached in all time steps. Usually, a simulation will not be canceled after one time step fails, but it will be continued based on a coarse approximation in the “hope” that the solver will recover within the following steps. As the sea ice problem is mostly data driven, with forcing from ocean, atmosphere and thermodynamics, this approach is reasonable.

The operator-related Newton scheme has no computational overhead. As it is able to accelerate the convergence it can reduce the required number of Newton iterations and therefore decrease the overall computational cost. Every change of the damping parameter $\delta$ will call for a new assembly of the Jacobian. This may limit the possible savings of simplified Newton methods.

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Figure 6: Inexact Newton method with different globalization and acceleration strategies under mesh refinement. Solving the test problem of Section 6, evaluating the number of Newton steps per time step. The relative nonlinear tolerance is set to $\text{tol}_r = 10^{-4}$, the global tolerance to $\text{tol}_g = 10^{-13}$. The simulation is terminated if the solver could not reduce the residual to the desired tolerance in 10 consecutive time steps.
Figure 7: Evaluating the damping parameter $\delta^{(l)}$, the reduction rate $Q_l$ and the residual of the operator-related damped Jacobian method at time step 160 on a 8 km, 4 km, 2 km mesh. The residual is normalized by the initial residual.
7.2. Analysis of the operator-related damped Jacobian method

Figure 6 shows that the standard Newton scheme with line search globalization (with or without a crude approximation of the linear systems) fails after 160 time steps on the 8km mesh. On finer meshes, these approaches fail even quicker. We therefore give an analysis of the convergence of the modified Newton solver in time step 160 to present the convergence behavior for this difficult case.

In Figure 7 we show the relation between the damping parameter $\delta^{(l)}$ and the reduction rate $Q_l$ (see (21)) of the operator-related damped Jacobian method for different mesh resolutions.

For large Newton reduction rates close to one, the damping parameter $\delta^{(l)}$ is decreased. Whenever the Newton convergence is good and reduction rates are small, $\delta^{(l)}$ is enlarged and chosen close to 1.

![Graph showing the relation between damping parameter and reduction rate for different mesh resolutions.](https://doi.org/10.1016/j.ocemod.2017.06.001)

Figure 8: Convergence of the full Newton scheme at time step 160 ($\delta^{(l)} = 1$ fixed). No convergence is observed for the meshes of size 4 km (top) and 2 km (bottom). Compare to Figure 7 for the operator-related Newton method.

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The operator-related damped Jacobian method is robust under mesh refinement. We obtain a reduction of the nonlinear residual on all mesh levels. The damping parameter \( \delta^{(l)} \) is not decreased on finer mesh, see Figure 7.

Although the parameter \( \delta^{(l)} \) is close to one in all iterations shown in Figure 7, a adaptive control of the Jacobian is necessary. A full Newton method with \( \delta^{(l)} \equiv 1 \) does not converge on the 4km and 2km mesh in this time step. For comparison, we show the behavior of the full Newton method in Figure 8.

Comparing the convergence of the Newton method in Figure 8 (without adaptive \( \delta \) control) to the lower two graphs of Figure 7 (including adaptive \( \delta \) control) shows that damping is necessary. Without the proper choice of the damping parameter \( \delta^{(l)} \) no convergence to the required tolerance is obtained. By an adaptive damping of the negative semidefinite part \( A'_2 \) we accelerate the Newton convergence. In some iterations, the full Newton without damping would not give convergence at all.

Finally, we present the damping of the operator-related damped Jacobian method in time step 280 on the 2km mesh, exemplary for a time step in which the solver needs a lot of iterations to converge, see Figure 9. One problem of Turek's original damping strategy is that too low values of \( \delta^{(l)} \) will limit the possible convergence to that of the simple Picard iteration. This results in reduction rates so close to one that the adaptive strategy (22) will never increase \( \delta^{(l)} \) again. Therefore, once \( \delta^{(l-1)} \) reaches the critical value of \( \delta^{(l-1)} < 0.2 \) we continue the procedure with \( \delta^{(l)} = 1 \). The idea is to restart the Newton scheme with a new initial value. The operator-related damped Jacobian method would not converge to the required tolerance within 200 iterations without limiting \( \delta^{(l)} \). Figure 9 shows a case, where \( \delta^{(l)} \) is reset to one twice.

Using the modified Newton solver with adaptive choice of \( \delta^{(l)} \), we can reduce the residual to a required tolerance in a limited number of iterations. Initially, \( \delta^{(l)} < 1 \) must be chosen to obtain convergence. Finally, transition to fast quadratic convergence is reached.

8. Conclusion

Based on a variational formulation of the sea ice equation we use finite elements as spatial discretization technique. We present and test a new modified Newton method, called the operator-related damped Jacobian method (see [33]), with a line search globalization. The idea of our approach is to combine a damped Newton solver with a Picard method.

Our modified Newton solver is based on the analytical evaluation of the Jacobian. We could show that the Jacobian of the VP model is a positive definite matrix. For positive definite Jacobians a damped Newton iteration will converge globally if a good damping parameter is chosen. The matrix is symmetric apart from the Coriolis term. Given an explicit treatment (like Zhang and Hibler [16] do) this will allow for simpler solvers like Conjugate Gradient [44]. Further we showed that the derivatives of the viscosities define a negative semidefinite matrix. By an adaptive controlling of the negative semidefinite part we can accelerate the convergence of the Newton iteration. In the regime of quadratic convergence rapid decay is reached by a transition to the full Newton scheme.

We compared our globalization strategy to the full Newton method with a line search damping and a crude approximation approach, which is used in the JFNK solver [27]. On a test problem we verify that our stabilization strategy gives better robustness. This is in particular true on fine meshes. In contrast to the latter two approaches, we could solve the test problem on a 8km mesh and a 4km mesh without failures. On a 2km mesh the inexact Newton method with the operator-related damped Jacobian method and line search globalization only fails 6 out of 350 times steps. Here we stress that “failure” just means, we did not reach the desired tolerance within a fixed number of iterations. On all mesh levels the other globalization strategies failed many times in a row without recovery, such that the problem can no longer be considered as “solved”. In an analysis we show that the operator-related

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Figure 9: Evaluating the residual of the operator-related damped Newton solver at time step 280, exemplary for a time step with severe convergence problems. The parameter $\delta(l)$ twice reaches the limit $\delta(l) < 0.2$ and is reset to $\delta(l) = 1$.

damped Jacobian method is robust under mesh refining. The damping parameter $\delta(l)$ is not chosen smaller on fine meshes. We suggest to test our globalization approach in simulations of the Arctic with realistic data.

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A. Proof of Theorem 1

All parts of the proof can be shown by basic calculus. We make frequent use of splitting the strain rate $\dot{\epsilon}$ (2) into deviatoric part and trace. For better readability we repeat the important definitions.

$$
\dot{\epsilon} = \dot{\epsilon} + \frac{1}{2} \text{tr}(\dot{\epsilon}) I, \quad \text{tr}(\dot{\epsilon}) = 0, \quad \text{tr}(\dot{\epsilon}) = \text{div} \mathbf{v},
$$

$$
\Delta(\dot{\epsilon}) = \sqrt{(2e^{-2}\dot{\epsilon} : \dot{\epsilon} + \text{tr}(\dot{\epsilon})^2 + \Delta_{\text{min}}^2},
$$

$$
A'(\mathbf{v})(\mathbf{w}, \phi) = A'_1(\mathbf{v})(\mathbf{w}, \phi) + A'_2(\mathbf{v})(\mathbf{w}, \phi),
$$

$$
A'_1(\mathbf{v})(\mathbf{w}, \phi) = (\rho_{\text{ice}} H \mathbf{w}, \phi) + k(f \cdot e_r \times \mathbf{w}, \phi) + k(\sigma'_1(\mathbf{w}), \nabla \phi) - k(\tau'_\text{ocean}(\mathbf{w}), \phi),
$$

$$
A'_2(\mathbf{v})(\mathbf{w}, \phi) = k(\sigma'_2(\mathbf{w}), \nabla \phi).
$$

(i) Splitting of the stresses $\sigma' = \sigma'_1 + \sigma'_2$ is organized such that $\sigma'_1$ includes the derivatives with respect to the strain rate $\dot{\epsilon}(\mathbf{v})$ while $\sigma'_2$ includes the derivatives with respect to the viscosities $\zeta(\dot{\epsilon})$ and $\eta(\dot{\epsilon})$.

Here, the fundamental expression is the derivative of $\Delta(\mathbf{v})^{-1}$, which is given as

$$
\frac{d\Delta^{-1}}{d\mathbf{v}}(\mathbf{w}) = -\frac{1}{\Delta^2} \frac{d\Delta}{d\mathbf{v}}(\mathbf{w}) = -\frac{1}{\Delta^2} \left(2e^{-2}\dot{\epsilon}'(\mathbf{w}) + \text{tr}(\dot{\epsilon}) \text{tr}(\mathbf{w}) \right).
$$

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All the remaining derivatives in (14) and (15) follow by basic calculus.

(ii) Symmetry. Next, we analyze the different terms in the Jacobian $A'(v)(w, \phi)$ for symmetry. As the triple product $(\ddot{a} \times \ddot{b}) \cdot \ddot{c} = -(\ddot{a} \times \ddot{c}) \cdot \ddot{b}$ is anticommutative, the Coriolis term will be non-symmetric. The term stemming from the time-derivative $(\rho_{\text{ice}} H w, \phi)$ is clearly symmetric. By $\mathbf{e}'(\cdot) = \mathbf{e}'(\cdot)^T$ we see that $\sigma'_1(w) = \sigma'_1(w)^T$ and $\sigma'_2(w) = \sigma'_2(w)^T$ are symmetric tensors, it holds
\[
(\sigma'(w), \nabla \phi) = \frac{1}{2} (\sigma'(w) + \sigma'(w)^T, \nabla \phi) = \frac{1}{2} (\sigma'(w), \nabla \phi + \nabla \phi^T) = (\dot{\sigma}'(w), \dot{\phi}(w)).
\]

Then, symmetry of $\sigma'_1$ follows by using the symmetry of the tensor product $\mathbf{e}'(w) : \mathbf{e}'(\phi) = \mathbf{e}'(\phi) : \mathbf{e}'(w)$ and $\mathbf{e}' : I = \text{tr}(\mathbf{e}') = 0$:
\[
(\sigma'_1(w), \nabla \phi) = \left[ 2e^{-2} \zeta \mathbf{e}'(w) + \zeta \text{tr}(\mathbf{e}(w)) I, \mathbf{e}'(\phi) + \frac{1}{2} \text{tr}(\mathbf{e}(\phi)) I \right] = \left[ 2e^{-2} \zeta \mathbf{e}'(\phi) + \zeta \text{tr}(\mathbf{e}(\phi)) \mathbf{I}, \mathbf{e}'(w) + \frac{1}{2} \text{tr}(\mathbf{e}(w)) \mathbf{I} \right] = (\sigma'_1(\phi), \dot{\mathbf{e}}(w)) = (\sigma'_1(\phi), \nabla \mathbf{w}).
\]

The second part $\sigma'_2$ is symmetric, as
\[
(\sigma'_2(w), \nabla \phi) = \left( \sigma'_2(w), \dot{\mathbf{e}}(\phi) + \frac{1}{2} \text{tr}(\mathbf{e}(\phi)) \mathbf{I} \right) = -\left( \frac{\zeta}{\Delta^2} \left[ 2e^{-2} \mathbf{e}' : \mathbf{e}'(w) + \text{tr}(\mathbf{e}) \text{tr}(\mathbf{e}(w)) \right] \left[ 2e^{-2} \mathbf{e}' + \text{tr}(\mathbf{e}) I \right], \mathbf{e}'(\phi) + \frac{1}{2} \text{tr}(\mathbf{e}(\phi)) I \right) = -\left( \frac{\zeta}{\Delta^2} \left[ 2e^{-2} \mathbf{e}' : \mathbf{e}'(w) + \text{tr}(\mathbf{e}) \text{tr}(\mathbf{e}(w)) \right], \left[ 2e^{-2} \mathbf{e}' : \mathbf{e}'(\phi) + \text{tr}(\mathbf{e}) \text{tr}(\mathbf{e}(\phi)) \right] \right).
\]

Likewise, symmetry holds for $\tau'_\text{ocean}$ given by (15). Finally, all terms except of the Coriolis term are symmetric.

(iii) Definiteness. To show definiteness of the Jacobian we first note that $(\mathbf{e}_r \times \mathbf{w}, \mathbf{w}) = 0$. Positive definiteness of $(\rho_{\text{ice}} H w, \mathbf{w})$ is obvious. The forcing term $-(\tau'_\text{ocean}(w), \mathbf{w})$ is positive semidefinite as
\[
-(\tau'_\text{ocean}(w), \mathbf{w}) = \rho_{\text{ocean}} C_{\text{ocean}} \left( \|v_{\text{ocean}} - \mathbf{v}\|^2 + \|v_{\text{ocean}} - \mathbf{v}\|\mathbf{w}\|^2 / \|v_{\text{ocean}} - \mathbf{v}\| \right).
\]

We show positive definiteness of $\sigma$. With (26) and (27) we obtain
\[
(\sigma'_1(w), \nabla \mathbf{w}) = \|\sqrt{2e^{-2}} \zeta \mathbf{e}'(w)\|^2 + \|\zeta \text{tr}(\mathbf{e}(w))\|^2, \quad (\sigma'_2(w), \nabla \mathbf{w}) = -\|\sqrt{\zeta / \Delta^2} \left[ 2e^{-2} \mathbf{e}' : \mathbf{e}'(w) + \text{tr}(\mathbf{e}) \text{tr}(\mathbf{e}(w)) \right] \|^2
\]
and observe positivity of $\sigma'_1$. To proceed with $\sigma'_2$ we define
\[
A := \sqrt{2e^{-2}} \mathbf{e}' + \frac{1}{\sqrt{2}} \text{tr}(\mathbf{e}) I, \quad B := \sqrt{2e^{-2}} \mathbf{e}'(w) + \frac{1}{\sqrt{2}} \text{tr}(\mathbf{e}(w)) I.
\]

By Cauchy-Schwarz inequality, it holds that $(\sum_{ij} A_{ij} B_{ij})^2 \leq \sum_{ij} A_{ij}^2 \sum_{ij} B_{ij}^2$. Using the notation of

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Combination of (28) and (30) with the triangle inequality gives

\[
(\sigma'_2(w), \nabla w) \geq - \max_{\Omega} \left\{ \frac{\Delta^2 - \Delta_{\min}^2}{\Delta^2} \right\} \left( \| \sqrt{2} e^{-2\epsilon} \sqrt{\epsilon'}(w) \|_2^2 + \| \sqrt{\zeta} \text{tr} (\epsilon(w)) \|_2^2 \right).
\]

Together with the estimate of \( \sigma'_1(w) \) in (28) we get

\[
(\sigma'(w), \nabla w) \geq \min_{\Omega} \left\{ \frac{\Delta_{\min}^2}{\Delta^2} \right\} \left( \| \sqrt{2} e^{-2\epsilon} \sqrt{\epsilon'}(w) \|_2^2 + \| \sqrt{\zeta} \text{tr} (\epsilon(w)) \|_2^2 \right) \geq c_K \min_{\Omega} \left\{ \frac{\Delta_{\min}^2}{\Delta^2} \right\} \| \sqrt{\zeta} w \|_{H^1(\Omega)}^2,
\]

where \( c_K > 0 \) is the constant of Korn’s inequality [5].

(iv) The upper bound is obtained in the same manner and by estimating

\[
|(\sigma'_2(w), \nabla w)| \leq \max_{\Omega} \left\{ \frac{\Delta^2 - \Delta_{\min}^2}{\Delta^2} \leq 1 \right\} \left( \| \sqrt{2} e^{-2\epsilon} \sqrt{\epsilon'}(w) \|_2^2 + \| \sqrt{\zeta} \text{tr} (\epsilon(w)) \|_2^2 \right).
\]

\[\square\]

References


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